THE
MATHEMATICAL GAZETTE

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THE PLACE OF VISUAL AIDS IN THE TEACHING OF MATHEMATICS.*

Mr. I. R. Vesselo: The problem of selecting a suitable programme of films and film strips for this meeting was not a very difficult one ; the choice is very limited. Like the evening-class system, which grew out of a desire, as much to make use of empty classrooms as to educate, the educational film grew partly out of a desire to make wider use of films whose first function was to fill out the normal entertainment programme. An audience would tolerate a little education whilst waiting for the " big picture ".

So we had biology films, geography films, history films, which by a slight adjustment, such as the addition of a more sophisticated commentary, could become animal films, travel films, or merely interest films; but not mathematical films, for this was too much to ask of a long-suffering audience. It was left to the enthusiasts—mathematicians with some technical knowledge of the film, or technicians with some mathematical knowledge, to produce the first mathematical films ; but not to the teacher. The hand of the teacher is absent.

We will begin with two films by Mr. Fairthorne. The first, the EQUATION $\ddot{x}+x=0$, is now well known ; in fact, it was shown as a sort of party piece at a recent exhibition for the general public. It is an attempt to bring the differential equation to life, and great credit is due to Mr. Fairthorne for his boldness and ingenuity.

It is based on the principle of the wheel and disc planimeter. A disc of radius $\ddot{x}$ feeds out a rigid string, whose length is $\dot{x}$. This regulates the radius of a second disc, which in turn feeds out a second rigid string, whose length is x. By linking up the second string with the first disc, we have the required relation. A large number of drawings of this system, for varying radii, were drawn and put together to give the impression of continuous motion. The discs become larger or smaller, positive (white) or negative (black) satisfying the equation all the while.

* A discussion and exhibition at the General Meeting of the Mathematical Association, 9th April, 1947. A list of exhibitors and helpers is given at the end of this report.

MATHEMATICS GRADUATE LIBRARY

There are three sequences. The first shows the symmetry of the relationship, as the motion of the two discs differs only in phase, and the periodic roots. In the second sequence there is superimposed the loci of the ends of the rotating radii. These are double circles whose diameters are equal to the maximum radius and whose axes of symmetry are at right angles. This brings out the quarter period difference in phase between velocity and displacement and the sine and cosine in the roots. In the last sequence are shown the harmonic curves which result if velocity and displacement are plotted against time.

If the film is run n times faster, it shows the equation $\ddot{x} + n^2x = 0$.

There is a second film in this notation, $\ddot{x} + x = a \sin nt$. In this film the lines representing acceleration and displacement are made to bear against the arms of a balance. The pivot is given a harmonic displacement of $a \sin nt$.

The second film, A HYPOCYCLIC MOTION, is rather longer and is, for the mathematician, crowded with incident. A rigid bar slides between two fixed straight lines. The instantaneous centre traces out a circle in the fixed plane and a second, of half the radius, rolling on the first on the plane carried by the moving bar. The rolling of the second circle on the first, internally, is shown to be equivalent to the original motion. Points on the circumference of the rolling circle trace out diameters of the fixed circle. This property has been used to obtain a pure S.H.M. and to obtain rotation from a linear reciprocating motion.

In the third section a point in the moving plane describes an ellipse. If the motion is generated, as above, by a bar slipping between two straight lines, the mechanism becomes elliptic trammels. If the bar is fixed and the straight lines move, a cutting tool in line with the bar will cut out an ellipse. This is the elliptic chuck. If the two straight lines are supposed to be rotating rigidly with shafts normal to the screen, passing through the ends of the bar, and a sliding crosspiece compels them to remain at right angles as in the film, we have Oldham's Coupling. Finally, the envelope of the diameter of the rolling circle is shown as the four-cusped hypocycloid.

Mr. Fairthorne has very kindly given a copy of this film to the Association. It is available on loan to members.

TRANSFER OF POWER, the third film, comes in a different category. Made by the Shell Film Unit, it is available for loan, free of charge, from the Petroleum Films Bureau. It is longer and has sound. It gives an account of the history of the toothed wheel, and deals with the epicycloid and involutes. As it supplies its own commentary, I will not attempt to compete with it ; I will only add that much more money was spent on it than on the previous films.

FILM STRIPS. The more recent development is the film strip. As few members will have seen them, I have included several in preference to the older films. There is a new technique involved here, full of possibilities. A series of still pictures are thrown on the screen, with suitable intervals for comment and development. Teaching notes are usually supplied, but they can, fortunately, be ignored.

To deal adequately with each strip would need more time than can be justified this afternoon, leaving little time for the papers and discussion to follow. I propose, therefore, to confine myself to generalities and to show the pictures at a rate which, whilst not allowing time for detailed study, will give a general idea of the scope of these film strips and what could be done if teachers were to indicate their desire for them.

LONG RANGE H.A. PROBLEM, I–V. This strip is in use in gunnery training in the Royal Navy and was lent by kindness of the Admiralty. It provides a real-life, or real-death, background to many problems in the ordinary school course. My own feeling is that this and similar film strips, drawn from the arts of peace, and used as the starting-point for many of our classroom topics

could bring to them the vigour which comes from appreciation, as distinct from technical skill.

AERODYNAMICS, I–II. This strip was lent by Common Ground, the producers. Although this firm and some others are experimenting with mathematical topics, there is no finished strip available from this source, although there are several sets of drawings, which may be seen in the exhibition upstairs. So I have included this strip on a near mathematical topic, to show the line of treatment.

GEOMETRICAL CONSTRUCTION, I–II. This is an American naval strip, used to teach, by mass production methods, the use of geometrical instruments. I do not suggest that it has any classroom significance. I have to thank Dr. Huppert, of the Scientific Film Association, for the loan of this strip.

Drawings from other strips in preparation, lent by Educational Publicity Ltd. and by Mr. Grattan-Guinness, are exhibited upstairs. Having taken up so much time on what is intended to be a prelude to the discussion, I will put my own views very briefly. Visual Aids have become fashionable and may bring much help to the teacher, or they may lead to much waste of energy, leaving nothing in their train, unless the voice of the teacher is loud and clear.

Confining myself to the film and film strip, I have set myself several questions and offered answers in the hope of stimulating discussion.

What has a film to offer to the teacher of mathematics? A film is expensive, in time and comfort as well as money, and it must compensate for all this if it is to be worth while. I think we must recognise that, perhaps as in music, appreciation is desirable, even without technical skill. For many pupils the film can supply the visual links without which much of their mathematics is meaningless.

The film can supply movement—the moving point really does move : it can give the appearance of depth, so that classroom geometry can emerge from "Flatland" into a three-dimensional world. But above all, it can give the whole picture in one broad sweep, bringing together members of one family, speeding up slow processes to show the completed product, as, for example, in making up a table. We have received a number of suggestions for possible films, and I hope we shall have more. Certainly the means of production is there, if the desire for it is voiced.

What makes a good film? Apart from technical qualities, it must add something which the teacher cannot already produce or cannot conveniently bring into the classroom.

Where is the film most needed? At the very bottom of the scale. Some of the films shown this afternoon are for the mathematician, to whom they are a luxury. But it is to the beginner, the average and the dull pupil, for whom they are a necessity, that they will make the difference between acquiring (or not) a meaningless technique and understanding a fascinating sequence of ideas.

How can they best be used? I think the naval film strip suggests the way, by leading off from a topic which is of practical interest, and drawing the mathematics from it. Of course, many teachers do this now, but the bounds of the school building are narrow.

A word about the new National Committee for Visual Aids in Education. I do not recognise, amongst its committees and sub-committees, a single mathematician. There is a danger that, because nothing useful has come out so far, it might be assumed that there is no scope for films or film strips in mathematics. So I hope that those who feel otherwise will say so.

I have left the discussion on models and charts to Mr. Meredith and those who follow. May I, however, take the opportunity, on behalf of the Visual

Aids sub-committee, to thank all those who have kindly lent models, charts, films, film strips or drawings, or who have so readily given their assistance.

Mr. G. Patrick Meredith (University College, Exeter): "Visual Aids" has become a very fashionable phrase recently. In talking about them one is nearly always expected to attend almost exclusively to apparatus, films, film strips and screens. I should like to draw the attention of the Mathematical Association to an inherent absurdity in most of the current propaganda on this subject. On the one hand we are urged on all sides to partake of the apocalyptic rewards of these marvellous and revolutionary devices, and are really led to suppose that something wonderful is happening in education. On the other hand, when we examine the content of the propaganda put out at conferences, in magazines, and on teachers' courses, what does it amount to? An endless re-hashing and elaboration of two facts :

1. That pictures and diagrams can be thrown on a screen.
2. That screen images can be made to move.

Now I submit that these two technical facts do not in themselves constitute an educational revolution, for children have had pictures and diagrams in their textbooks for generations. The mere fact of putting them on a screen instead makes no essential difference—*unless* (and this is the crux) unless some new principle is at work in the selection or design of the pictures and diagrams. And I further submit that the current mania for gadgetry of all kinds does not offer a favourable atmosphere for the emergence of such a new principle. On the contrary, experience has taught me that unless one prattles away *ad nauseam* about bloomed lenses, P.E. cells, translucent screens, rotating heads, click feeds and fluff at the gate nobody listens. I am therefore reluctantly coming to the conclusion that Visual Aids in their present phase are largely a fraud. Don't misunderstand me. To introduce motion and bright images into the classroom is a very desirable improvement—along with all the other educational improvements—better buildings, school milk, the raising of the school-leaving age, etc. But the claims made for Visual Aids go far beyond this. Yet no one seems to be really interested in finding out how to make them fulfil their promise. The answer does not lie in mere mechanism.

It may well be that the Mathematical Association is the one body whose members can really see the point. I say this because they know a mathematical revolution when they see one. And it was mathematics which made the most important revolution in visual expression since the invention of the alphabet. I mean Cartesian Geometry. When you pause to think of the endless consequences, both mathematical and practical, which have flowed from Descartes' device for giving spatial (and therefore *visual*) expression to algebraic properties, I think you will agree that this truly was a revolution.

The point to which I am leading is that we urgently need another advance of comparable, or even greater importance. And I believe it is on the way. It will be both a mathematical and a visual advance. And it will owe little or nothing to the efforts of most of the present assiduous propagandists for Visual Aids, for the simple reason that they are not even looking for it.

I have said it will be a mathematical advance, but I am taking mathematics in its widest modern sense, to include logic. I have time only for a few words in support of this prophecy.

Consider the problem of Visual Education in its simplest terms. I take Visual Education to mean not simply the showing of pictures, but *mental development resulting from material presented to the eye.* And the kind of mental development which mathematicians are interested in is in the direction of clear thinking. Hitherto we have been content to distinguish between two

types of facts, viz. : quantitative and qualitative. It has been the function of mathematicians to develop precise mental habits in the handling of quantitative problems and to leave the qualitative ones to others. But to-day, with such developments as symbolic logic and topology coming within the scope of mathematics we are beginning to realise that number and geometric form do not exhaust the possibilities of exact reasoning. For example, if two closed plane curves intersect, their size and shape have nothing to do with the fact that they divide the plane into four regions about which a number of precise assertions can be made. And when we look mathematically at a screen image, we see that it consists essentially of a number of regions differing in brightness and colour. It is therefore capable of expressing all manner of logical relations—of inclusion, exclusion, opposition, equality, and so on. Thus the basis of Visual Education is essentially mathematical. Conversely it might be said that all mathematics, other than oral and mental calculations, is visual. It is represented by visible signs on paper and taken in by the eye. But the word "visual" has recently acquired a new emphasis. First, let me describe any visible representation, whether geometric, photographic, algebraic, numerical, verbal, etc., as a "graphic". Graphics are designed to give expression either to concrete facts or to mental concepts. Every fact or concept has certain important aspects, and a host of unimportant aspects. Graphics are often drawn which are not in any sense incorrect, and which yet fail to distinguish either the important aspects from the unimportant, or one important aspect from another. *A "visual" graphic, in the new sense, is one which clearly makes these distinctions.* I shall call a visual graphic, so characterised, a "semagraph" (Greek, $\sigma\eta\mu\alpha$ = a sign). In a semagraph every important dot, line, area or volume is designed to give a clear and distinct meaning. Thus there must be as many visually distinct elements in the semagraph as there are important and distinguishable aspects in the fact or concept to be represented.

There is a further and more subtle requirement. The brain of the reader does not passively receive impressions from a graphic. It actively interprets them and extracts all manner of further inferences not explicitly intended. The semagraphic expert needs great skill and insight in order to create designs in which no false inferences are implicit in the symbolism adopted. The symbolism should be vivid, and yet make no suggestions of its own inconsistent with the logic of the original fact or concept. *Thus the whole symbolism must be designed in its colours, shapes, proportions and details so as to be consonant with the quality and structure of the fact or concept which it represents.*

With this modern interpretation of the term "visual" it is evident that many of the illustrations in mathematical textbooks of the past do not qualify as "visual" representations. They are *graphic*, but not *semagraphic*.

The mathematician naturally tends to feel that when he has given algebraic expression to some abstract truth he has done his duty. The handwriting or print, the colour of the ink, the quality of paper, the size and lay-out of the page, are from the mathematical point of view irrelevant. But in so far as he is trying to communicate his proposition to someone else, these things are extremely important. To tell the truth is not enough. It must be *communicated*. If the house is on fire, we do not do our duty by merely saying "fire". We shout "Fire! Fire!" Colour, contrast and arrangement are the visual equivalents of intonation in the voice, and we know how meaning can be radically altered by a change of intonation.

Visual communication has a great advantage over oral communication in that it has two or three spatial dimensions at its service, instead of the single time-dimension of the voice. It can thus make evident a vastly greater number of relationships. Relationship plays an important part in modern logic. Indeed, any systematically taught school subject is essentially an

exposition of particular types of relationship—of man to man, of matter to energy, of climate to products, and so on. Thus if the mathematical basis of the arrangement of patches of light on a screen, to convey educational information, can be satisfactorily worked out, the whole curriculum will benefit. Such a basis would constitute an even greater revolution than Cartesian Geometry. Indeed, Cartesian Geometry is merely a special case of a semagraph. For there the form of a line is so drawn as to express an algebraic relationship between two or more variables.

The Figurate Arithmetic, or Geometry of Numbers, about which Professor Davenport is shortly going to speak, and which I also happen to have illustrated by some models in the exhibition, is another special case of visual elements being used to express logical relationships. Our whole exhibition is a testimony to the power of visual expression. In these two rooms and in these two days we have an array of solid ideas exceeding anything to be found in the biggest film studio in ten years. But we have not yet achieved a neo-Cartesian synthesis. The basic forms of these models and charts are, of course, dictated by the subject-matter, but the colour, size and arrangement are matters largely of intuition. The revolution in Visual Education will come when we can design any piece of visual teaching material—chart, model or film—with the same consistency, certainty of communication and lack of arbitrariness as we enjoy with the ordinary Cartesian graph.

The energy and talent indicated in this exhibition of Visual Aids suggest that the big step forward may come quite soon.

Mr. J. Dickson (Coventry Technical College) pointed out that an individual who was trying to visualise a mathematical idea always had some image in his mind ; it seemed that the use of the films, or film strips, would dictate to the minds of students a method of visualising mathematics which might not prove altogether suitable to a particular student. It would probably be said that the textbook diagram was also unsuitable to all types of minds, but the textbook diagram was two-dimensional, whereas he (the speaker) was thinking of three-dimensional or even four-dimensional problems which each individual visualised for himself ; the use of a film strip or a film which could give some idea of three dimensions might put an end to further imagination on the part of an individual.

Miss Moore (Midleton College, Purley) was reminded by the last speaker's remarks of the psychologists' discovery of the working of people's minds in connection with numbers ; many people had what were termed " number forms " ; when they thought of numbers they visualised them as going in definite directions. Being interested in that problem when a student, the speaker had collected instances of such visualisation. The speaker had found that she herself had certain number forms, and it had seemed to her that it was the slow-working mathematicians rather than the rapid workers who had such number forms. They found it helped them to remember numbers they had been told, but when one put any sort of form on a blackboard which did not happen to be the particular number form a person had in mind, it tended to muddle that person.

Mr. A. K. French (Stroud) had been much interested in the subject of which Mr. Meredith had spoken, and he thought he understood what that speaker had in mind when he spoke of the need for a new revolution in the presentation of visual material. The difficulty in the past had arisen from the fact that it had been left to mathematicians to write mathematics textbooks. That might sound peculiar, but the problems of visual presentation lay partly in the realm of other branches of philosophy, or science, those of the psychologist and philosopher, and partly in the realm of the artist. It seemed, therefore, that the necessary change in the presentation of visual material

must come from a cooperative effort. It would not be sufficient for one or two individuals to sit down to write a mathematics textbook ; there was need for a group of experts, each in their own branch of art or science, to come together—the artist and the scientist, as well as the mathematician. If there could be team work on that basis, then it would result in the production of a more suitable type of textbook. Something on those lines had already been applied by certain groups of people to the production of books for adults. It was not, however, of much use to produce such books for adults who had, in the first instance, been taught by means of such poor material as that generally available. Mr. French looked forward to the time when there was no longer a shortage of paper, and it would be possible to have the technique he had mentioned applied to school textbooks as well as to books for adults.

Mr. N. R. Eyres (Radley) felt that Mr. Meredith had made a rather slighting reference to " the age of gadgetry ". Was he condemning all visual aids? The speaker added that he had had the good fortune during the war years to work on the Differential Analyser with Professor Hartree at Manchester University, and he had come to the conclusion that it was one of the best visual aids that could be imagined. It was a machine for the solution of differential equations ; it set up mechanically the whole of the differential equation and the process of solving it in terms of shafts and gears, and as he personally had the sort of mind that enjoyed studying shafts and gears he had found that the use of the Differential Analyser had enormously increased his appreciation of what was really going on in the solution of an equation. Each quantity appearing in the equation was represented by a shaft ; the size of the quantity at any given moment was represented by the number of rotations the shaft had made. The film solution of $\ddot{x}+x=0$ had reminded the speaker very much of the process going on in the solution of the same equation on the Differential Analyser. As a result of his experience with the Differential Analyser he was strongly in favour of all forms of visual aid.

Mr. P. D. Greenall (Scientific Civil Service) felt that the value of visual aids in mathematical teaching should not remain a matter only of opinion ; such questions should be settled on the lines of " the proof of the pudding is in the eating ". He wondered whether there was any experimental work which enabled one to know whether or not improvement in mathematical teaching resulted from the more extensive use of various visual aids. Had any experiments been carried out in which to similar groups had been taught with and without an aid, a control and an experimental group, and their results studied at the end of the period of learning to ascertain whether one group had understood and remembered more of the subject than had the other group?

Mr. J. W. Ashley Smith (Henry Smith School, Hartlepool) said that in teaching geography in his youth he had used episcopes and such aids—he did not use them now that he was a whole-time mathematician—and he had found them useful, but in geography one used all kinds of pictorial representations with the object of enabling the pupils to understand and remember things with which they could not have any personal contact. That, however, did not appear to be what one wished to do when teaching mathematics, because in teaching that subject it was necessary to avoid inducing pupils to conceive the material of the subject as things with which they did not have personal contact. It was probably detrimental to use for the purpose of representing mathematics anything which the pupils could not themselves reproduce. Far better to have a clumsy piece of apparatus made out of the top of a desk and a boy's head and a ruler to represent solid figures than to have resort to any visual aid the children could not themselves produce.

Mr. S. E. Cruise (King's School, Macclesfield) pointed out that no speaker had mentioned or stressed the need for increasing the use of different colours in visual aids. Some time ago there was a game in which one had a large number of counters on which there were figures, including fractions. The figures had to be arranged in groups of three which were alike ; following that, one did exactly the same thing with colours. The players were timed and, strangely enough, one did not discover that one had two hands with which to do the job until one came to the coloured counters, and in every case they were arranged very quickly. That game had an application which had been used with some success in the teaching of elementary geometry, particularly constructions. A friend had tried it on boys who had said that they were no good at geometry and would never learn any, but in their case the method was to draw all the original lines in black, the construction lines in green and the final line in red. Furthermore, the order in which the construction lines were made was denoted by dotted lines. That type of visual aid was exceedingly useful, and it did not seem that sufficient use was made of it at the present time.

Mr. L. R. B. Ehrenberg (St. Bees) thought many visual aids had come into use as a result of training men in the Forces ; largely because the instructors were not sufficiently good to put the information over by themselves. Such a film strip as that describing how to aim at a moving aeroplane had for its object the explanation of something which could easily be demonstrated by means of a piece of chalk and a blackboard, and probably to better effect because the teacher could use his individual method instead of being tied to that handed him on the film strip. There seemed, in fact, little point in that type of film strip being introduced into schools. The speaker added that he felt somewhat strongly on that point, because he had attended three conferences of teachers within the last year and seen such aids at each one of them, and he had come to the conclusion that those who were trying to persuade teachers to use visual aids in the form of film strips were often wasting the time of the teachers by even showing the strips to them. This particular type of film strip was useless. Even if it were used in a school, the teacher would need to go to and fro from the projector to the board to point out the various details on the diagram and would have to go back to change the diagram unless he had an assistant to do that. Also as soon as a diagram had gone the teacher could not conveniently get it back, whereas in blackboard work one could have three or four successive diagrams on the board at the same time ; with the film strip each had to be taken in turn. The particular type of film strip shown seemed to be a way out of doing any blackboard work. When it came to putting on a screen things that could not possibly be put on a blackboard it was a totally different matter, but there appeared to be little of that type of thing needed in the teaching of mathematics. As had already been pointed out, geography was a subject far more suitable for that type of matter ; certainly the speaker could not imagine many instances in which it would be really helpful in the teaching of mathematics.

Mr. A. R. Pargeter (Taunton's School, Southampton) asked were those taking part in the discussion interested primarily in visual aids in teaching or in learning? When actually teaching it might be much more fun and more impressive to produce some beautiful model, film or whatever it might be, than to try to get the teaching across by means of the blackboard and chalk, perhaps not very satisfactorily. But was it, perhaps, not more important to encourage pupils to get for themselves a better understanding of what they were learning by encouraging them to make their own diagrams and models out of " sealing wax and string " or whatever was available? In other words, was it not more important to emphasise the use of visual aids as a means of

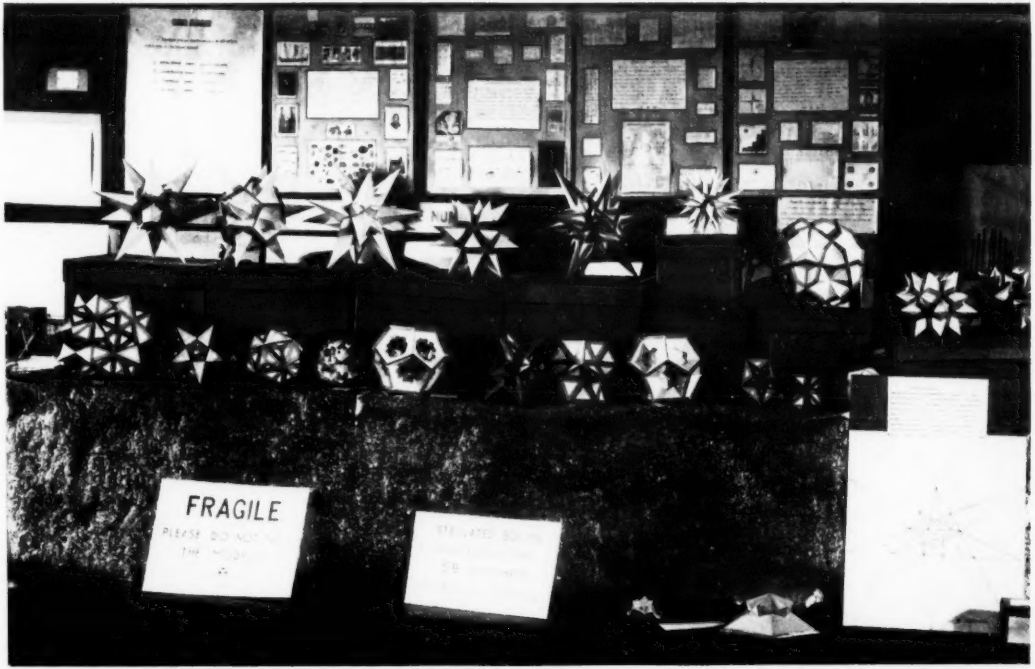


PLATE II. VISUAL AIDS EXHIBITION, 1947.

PLATE III.

VISUAL ARTS EXHIBITION, 1917.

pupils teaching themselves than to provide teachers with ready-made aids wherewith to convey their teaching?

Mr. A. W. Siddons (formerly Harrow School) said that in his early days he had used a number of coloured sticks with needle points at the end, made by Cussens of Manchester. When teaching he had used those coloured sticks to build up models of three-dimensional figures. Having built up a model he would put it on the desk and tell the pupils to make their own drawing of it. Then he would walk round and criticise the drawings. In the next lesson he would again build up the model, tell the pupils to take a good look at it, then dismantle it and ask them to make their own drawing of what they had seen. In the following lesson he described the model, and then told the children to make their drawings without the model having been in front of them. In such a case he was using a visual aid to begin with and developing the pupils' power to reproduce the figure without the model before them, and he had always regarded that as a valuable piece of education.

Mr. A. H. Hall (Risley Training College) thought it well to keep in mind the word " aid " when speaking of visual aids. It was unwise to decry visual aids because those seen or experienced were not suitable for one's own use, or because they appealed more to the teacher as a lazy person's aid than to the child as a means of learning. All could think of many textbooks of which the same might be said, but books were chosen which were regarded as likely to be of most use. So it should be with visual aids. In the matter of film strips and films there should be sufficient choice to enable teachers to use those which appealed to them, and those which they felt would help them to teach and help their particular pupils to learn. The Mathematical Association as such should be the first to sponsor such films and film strips. That was one of the functions of such a body.

In this connection the speaker commended two further points : One, that there should be an endeavour to acquire some films of historical interest depicting the development of the history of mathematics throughout the centuries. Children needed some background of that nature. Secondly, there was the point that children grew up to be very much afraid of mathematics, though those who were mathematicians probably did not appreciate that as much as they might. At the moment the speaker was concerned with emergency-trained teachers, and he could see amongst those men and women who had come from service in the Forces a great deal of disturbance and fear in regard to mathematics ; they only needed reassurance and sympathetic handling, together with interest, to start them on a new mathematical career. Mr. Hall felt sure that if there were an endeavour to go out to teach mathematics in the right way to youngsters, it would be possible to make on them an impression which would last for all time. In another three years there would probably be county colleges in existence, and pupils leaving school would be expected to go on with mathematics ; it was desirable that they should go on with the subject feeling interested and eager to learn. It was necessary that they should develop an interest in mathematics.

Finally, when speaking of visual aids it was necessary also to think in terms of visual education. It was vital for pupils and adults to appreciate the beauty they could make and could find all around them ; to realise that whilst there was beauty in a straight line there was not so much beauty in straight lines crossed at right-angles which one saw from the air in slum districts ; it was necessary for there to be appreciation of beauty of design. Pupils could be taken back again through historical films to the beauties of past architectures. In conclusion, Mr. Hall appealed again to the Mathematical Association to take a lead in trying to develop that type of aesthetic appreciation ; it should not be divorced from mathematics and the arts, but linked to mathematics.

Visual Aids Exhibition, 1947.
Plate III.

There being no further comments, The **President (Mr. W. F. Bushell)** called upon Mr. Meredith to reply to some of the points raised, whereupon :

Mr. G. Patrick Meredith, in reply to Mr. Dickson, pointed out that actually dictation had been used in education for a long time past ; in fact, it might be said to be part of the business of the school to mediate the dictation of the real objective world to the subjective world of the child, the point being that reality did dictate to mankind ; they could not prescribe what the world should be like, and thus there were many things in education that *had* to be done ; if passing those things on to the child was " dictation ", then there must be dictation. Secondly, there were many things which while not in the same sense objective were necessarily conventional, such as the alphabet, forms of numbers, rules of the road, weights and measures, and so on, all agreed on by social convention. There still remained a large measure of freedom which allowed the teacher to give the child his head to imagine whatever forms he liked. That was a topic which might well be explored further. There was, in any event, need to consider those three areas : the partly subjective area of free imagination, the conventionally controlled area, and the objectively-dictated area.

As to " gadgetry ", of course he was being provocative in that regard. He was all for gadgets of a kind, but in their place. The kind of gadget that had been described, the Differential Analyser, was a gadget of which every part had a mathematical function to perform, and it was an extremely valuable gadget if one might use the term. But what had been worrying the speaker in visual education—and he had been to many teachers' courses devoted to visual education—was the amount of time spent by teachers in worrying about all the finicky details of projectors, and so on. He would not mind teachers spending their time on such things if at the same time they showed the same degree of interest in what was being put on the screen ; but they did not. Therefore, Mr. Meredith uttered his warning against an excessive interest in gadgetry for its own sake.

He had not previously made the point—and there was not much time to refer to it on that occasion—that in all education the most important thing was the child's own activity. Whatever teachers did in visual presentation must be closely related to the child's own activity and be a stimulus to further activity. Any visual aid which led the child on to further exploration on his own was exceedingly valuable. The reference by Mr. Siddons to the simple method of leading, step by step, from the concrete to the abstract was a most helpful hint.

With regard to the remark as to the difficulty of quick reference back to an earlier diagram on a film strip, he personally would have thought the objection was the other way round : that when one rubbed a diagram off a blackboard it was gone, and there was the labour of re-drawing it if it was needed for further reference. Many film strips were reversible, and it was not difficult to turn back to repeat any given picture.

As to the use of colour in geometry, there was in the Exhibition upstairs a book by Oliver Byrne, about one hundred years old, from the library of the Mathematical Association, which might usefully be referred to. Students of the speaker's, in tackling the colour presentation of geometry and working according to their own ideas, had come on exactly the same method as that described in Byrne's book, and they would probably feel disappointed when they realised that they had been antedated by one hundred years.

Finally, there had been a number of experiments comparing visual methods with other methods, but the speaker feared that the statistical methods employed in analysing the results of the experiments had not been all that they might have been. He personally would not rely solely on experimental

elements. After all, the human race did not carry out experiments with the alphabet before using it ; the alphabet was invented and then used.

Mr. W. W. Sawyer, in summing up the discussion, said that Mr. Meredith had perhaps made his opening attack on " gadgets " in order to provoke discussion. He was sorry Mr. Meredith had partially recanted this attack, as he had spent a lot of time thinking of rude things to say in reply.

The points made by those who took part in the discussion were very sound. The danger of the film dictating to the child what it should do was an important point. Incidentally, it was a curious fact that modern children were criticised for passively sitting in cinemas instead of amusing themselves, but nobody seemed to have noticed that there was a much older institution—the traditional school—in which children had to sit still and absorb whatever was thrust at them.

In regard to Mr. Ashley Smith's point, the speaker had seen a good deal of work done in Leicester on the construction of apparatus. There were two factions there—the Heath-Robinson faction (to which he himself belonged) which held that apparatus should be comparatively primitive, so that the pupils were more likely to make it themselves at home ; and the faction led by Mr. Collins of the Gateway which maintained that everything must be beautifully finished and have a high aesthetic appeal. Both schools of thought were right. Both classes of model had their own type of beauty. The great value of Mr. Collins' work was in his skilful use of colour, which appealed strongly to children.

Another important point made in discussion was that the models should be made by the children themselves. This was very true. Teachers frequently found that in designing and making a model they themselves learnt far more than the class did from looking at the finished model. The speaker had tried the experiment of telling his class to design some kind of visual aid, and teach some point with its help. This was an interesting experiment, because it revealed how far the pupils' capacity for sorting things out had gone. He himself was never content simply to know a theorem or a result. He would keep turning it over in his mind until it reached a form in which it was immediately obvious. One could not always achieve this goal, but it was something to work towards. It was good for pupils to acquire this habit of trying to make their knowledge what the Germans call *anschaulich*—not just to know that something was true, but to see it as being naturally and rightly so.

The fear of mathematics was strong, and the important point had been raised that there were students in Training Colleges learning to teach mathematics while themselves desperately afraid of the subject, and having not the least idea what it was all about. And, of course, there were many fully-fledged teachers in the same plight. Other teachers did not seem to feel that the children would ever understand mathematics, and so they gave a number of rules which were to be followed blindly. For instance, in dealing with trigonometric identities " you must not mix the two sides together ; you must first work out one side and then the other "—this rule was applied blindly even in examples where a term on one side was obviously longing to mate with a term on the other side of the equals sign. The only rule he allowed in his classes was : " If it works, it is right."

Mr. Meredith had made some disparaging remarks about whether visual aids and gadgets really constituted a revolution in teaching. Of course, the speaker did not claim that any one procedure was a cure-all for educational difficulties. As he saw it, mathematics teaching was a sort of all-in wrestling. One had to cope with the most varied problems. He himself had developed the use of gadgets when teaching engineering apprentices. The lads understood machinery, and if you could gradually extract mathematical truths in

the course of studying some mechanism, they did not realise that it was mathematics and so they were not frightened in advance. So this method met that particular situation. Later the speaker had been called on, during an emergency, to teach School Certificate Geometry in a Girls' Grammar School. That was an entirely different situation. Only a minority of the girls were interested in mechanisms. New methods had to be found. As a zeroth approximation, one could say all boys were mechanics and all girls were artists. In the Exhibition there was a book of Euclid in which lines and angles were not named, but were marked with distinctive colours. He had given girls the job of translating theorems into colour, and they had been interested in this. There was also in the Exhibition an excellent example of the needlework approach to geometry ; this, too, appealed to girls.

One sometimes tended to think of diagrams and models as having simply an intellectual purpose—to make some point clear. But apparatus could also be used for an even more important emotional purpose—for creating the will to know. Even the clearest diagram was useless if a student did not want to know.

This was one of the weaknesses in the teaching of algebra. Algebra was a shorthand in which mathematical truths could be expressed. But suppose you had no truths within you aching for expression—what good was algebra to you then? It seemed more logical first to let students discover by simple experiments various scientific and mathematical laws, and then to encourage them to use abbreviations to lessen the labour of writing these laws down. In this way, one might succeed in enabling them to re-traverse the historical road by which algebra was originally invented. Then they might have some understanding of algebra. Too often, with the approach of the older textbooks, pupils had not the faintest idea why some algebraic equations were correct and others were wrong.

Mr. Meredith had laid great stress on the importance of system. System was all right in its way, but probably the reason why men were organic creatures and not machines was that life was infinitely varied ; one had perpetually to improvise. The speaker doubted if the revolution in teaching would be traceable to any *new idea*. There had always been first-class teachers, though only a few. Socrates had been such a teacher of mathematics—as, for instance, in the *Phaedo*, where he enabled a slave to discover a particular case of Pythagoras' Theorem for himself. Montaigne's views of education in the sixteenth century were hard-hitting, and he would still be a pioneer if he were alive to-day. The revolution in teaching would come if instead of having a few we had a great number of really live teachers, teachers who understood what they were doing, who liked mathematics and could somehow communicate that liking to others. That the teacher should be alive and that he should respect and encourage the vitality of his pupils—that was the essential thing. Gadgets and visual aids were merely a means to an end, to be used or not as occasion demanded—but often a very useful accessory.

The **President** felt sure all present would wish to voice their grateful thanks to the readers of the papers, though he had not that so much in mind as the work they had done, particularly Mr. Vesselo, in organising the very striking Exhibition to which many references had been made, and which many had already so much enjoyed seeing. There were some fifty to sixty exhibits, and schools and other places of learning had sent material in. All would realise that that was not done without trouble ; it had, indeed, required much organisation. To the President's personal knowledge Mr. Vesselo had written a vast number of letters and taken a great deal of trouble to collect the exhibits together and to organise the whole of the Exhibition in an extremely able manner. For that particularly novel procedure in the history of the

Mathematical Association all members were particularly indebted to Mr. Vesselo, and the President was himself most grateful.

VISUAL AIDS EXHIBITION.

1. *Polyhedra and models of quadric surfaces.*
J. H. Cadwell; Dr. H. M. Cundy, whose home-made harmonograph attracted much attention; H. Flather; M. I. Hemstead; Dr. J. C. P. Miller, whose exhibits included a book of explanatory drawings; Dr. J. G. Walker.

2. *Working models and demonstrations.*
The chief exhibits in this section were a group from Leicester, arranged by W. W. Sawyer, with the assistance of R. H. Collins, W. E. Date, C. L. Johnson, R. Kitchen and K. F. Solloway.
Other exhibitors in this section included P. F. Burns, C. W. Hansel, C. V. J. Hart, A. R. Kelsey, Miss M. J. Meetham, L. A. Swinden, Dr. J. Topping.

3. *Charts and maps.*
P. C. Davy, R. A. Fairthorne, Ford Motor Co., Isotype Institute, London Museum, Dr. S. Lilley, G. P. Meredith, Sculthorps Ltd., I. R. Vesselo, Miss M. R. Walker.

4. *Pictures and designs.*
Miss L. E. Hardcastle, J. J. Higgins.

5. *Primary school apparatus.*
Miss P. Child, Montessori Centre, J. W. Smith, W. Sweetman, Mrs. Thornhill, N. F. R. Watts.

6. *Mixed collections.*
W. F. Bushell, W. Hope-Jones, W. J. Langford, Science Museum, Dr. F. G. Maunsell, Miss A. M. Trout.

7. *Other items.*
A. R. Bielby (cross-stave), Professor H. R. Hassé (Klein bottle), Professor W. H. McCrea (magic cube), Professor E. H. Neville (books).

8. *Films, film-strips and drawings.*
The Admiralty, Common Ground Ltd., Educational Publicity Ltd., G. H. Grattan-Guinness, Dr. E. Huppert, J. Maddison.

9. Thanks are also due to the following: Mr. Ashley and his assistants for their help in arranging and despatching exhibits; to the British Film Institute for the loan of projectors and operator; to the gallant band of mathematicians-to-be, Messrs. Budd, Cooper, Hollings, Lanzer and Wilson, who gave up three days of holidays to act as stewards; and to the many exhibitors and others who assisted in countless ways.

THE GEOMETRY OF NUMBERS.*

By H. Davenport.

I would like first to thank the Association for the honour they have done me in inviting me to lecture at this meeting. It is, however, an honour which carries with it a certain danger. When a mathematician comes to lecture on a subject in which he has specialised, he is always liable to suffer from the delusion that what has gradually become plain and straightforward to him will at once appear plain and straightforward to others. It may be that I am suffering from this delusion in thinking that I can present the main idea of the geometry of numbers in one lecture.

The theory of numbers is concerned primarily with properties of the integers, or of the rational numbers. It is a remarkable fact that many such properties are most conveniently approached by methods which use irrational numbers, or again by methods which use continuous variables. Also equations in integers can often be best approached from the corresponding inequalities. As a simple example, suppose we wish to prove that the equation

$$x^2 - 2y^2 = \pm 1$$

has an infinity of integral solutions. One way of doing so would be to prove that the inequality

$$| x^2 - 2y^2 | < 2$$

has an infinity of integral solutions; and on factorising this we find that it would suffice to prove that there are an infinity of fractions x/y for which

$$\left| \frac{x}{y} - \sqrt{2} \right| < \frac{c}{y^2},$$

for some constant c less than $1/\sqrt{2}$. Thus a problem of a purely arithmetical character is reduced to a problem of *Diophantine approximation*. True, in this particularly simple example, it is hardly worth while to do so, because it is easier to treat the equation directly; but the principle is one of great generality.

The general problem of Diophantine approximation may be stated as follows. Let $f(x_1, \ldots, x_n)$ be a function of n variables, which is always positive or zero. How small can we make $f(x_1, \ldots, x_n)$ for integral values of the variables? If f is homogeneous, we must exclude the trivial possibility $x_1 = \ldots = x_n = 0$ which makes $f = 0$. In this general form, the question is necessarily somewhat vague, since one naturally asks, how small, in terms of what? When f is an algebraic form, or a combination of such forms, the answer is that one aims at an inequality in terms of the algebraic invariant, or invariants, of f. The problem of Diophantine approximation is of a mixed arithmetic and analytic nature, since the variables are integers and the coefficients of f are supposed to be arbitrary real numbers.

The geometry of numbers is an approach to problems of Diophantine approximation, suggested by interpreting them geometrically. The inequality $f(x_1, \ldots, x_n) < \lambda$ represents a certain region in n dimensional space. Under what conditions does this region contain a point with integral coordinates? The origin O, if it obviously belongs to the region, is to be excluded from consideration.

It was one particular problem of this kind which led Minkowski, in 1891, to his general theorems in the geometry of numbers. Take $f(x_1, \ldots, x_n)$ to be a quadratic form,

$$f(x_1, \ldots, x_n) = a_{11}x_1^2 + \ldots + 2a_{12}x_1x_2 + \ldots,$$

* Lecture delivered on 9th April, 1947, to the General Meeting of the Mathematical Association.

which is positive definite, i.e. strictly positive except when $x_1, \ldots, x_n$ are all zero. In this case, the inequality $f(x_1, \ldots, x_n) < \lambda$ represents the interior of an ellipsoid in n dimensional space, of a given shape and orientation, which expands as λ increases. The problem now is: how far must we expand the ellipsoid in order to be certain that it contains a point with integral coordinates, other than O? One possible answer * that suggests itself is that it may suffice to expand until the volume of the ellipsoid is greater than some constant (depending only on n). This is in fact true, but it is not immediately obvious. One may think it possible for an ellipsoid to have a large volume, but to be very long and thin, and to stretch out in some direction which enables it to avoid all the points with integral coordinates.

The theorem I have mentioned, namely that there exists a constant (depending only on n) such that any ellipsoid with centre O whose volume exceeds that constant necessarily contains a point other than O with integral coordinates, was first proved by Hermite in 1845. He did not use this geometrical interpretation, but considered the problem in its original formulation, as that of the minimum of a positive definite quadratic form, and proved the result by induction on n. The merit of Minkowski's proof lies in its simplicity and in its far greater generality and power.

Minkowski's argument was as follows. Let E be an ellipsoid, with centre O, which does not contain, in its interior, any point with integral coordinates except O. Let $\frac{1}{2}E$ denote the ellipsoid obtained by contracting E to half the linear dimensions. Place ellipsoids congruent † to the ellipsoid $\frac{1}{2}E$ with their centres at all points with integral coordinates. Then no two of this system of congruent ellipsoids can overlap. For if they did, then by the regularity of their distribution there would be one, with centre P, say, which overlapped the one with centre O. Then the mid-point of OP would lie in $\frac{1}{2}E$, and P would lie in E, contrary to hypothesis.

Thus we have a system of non-intersecting ellipsoids, all congruent to $\frac{1}{2}E$, centred at all points with integral coordinates. Let V denote the volume of E; then the volume of $\frac{1}{2}E$ is $V/2^n$. Intuition suggests that, in view of the non-intersection, the volume of $\frac{1}{2}E$ cannot exceed 1, and this is easily proved by a rigorous argument. Consider those ellipsoids of the system whose centres $(x_1, \ldots, x_n)$ satisfy

$$0 \leqslant x_1 < N, \ldots, \quad 0 \leqslant x_n < N,$$

where N is a large positive integer. Their number is N^n, their total volume is $N^n V/2^n$. They do not overlap, and they all lie in the cube

$$-k \leqslant x_1 < N + k, \ldots, \quad -k \leqslant x_n < N + k,$$

where k depends only on the ellipsoid, and not on N. (k is simply the greatest numerical value of any coordinate of a point on the surface of the ellipsoid $\frac{1}{2}E$.) Thus

$$N^n V/2^n \leqslant (N + 2k)^n,$$

whence

$$V \leqslant 2^n (1 + 2k/N)^n.$$

Since N is arbitrarily large, this implies $V \leqslant 2^n$.

We have therefore proved that *an ellipsoid with centre O, which does not contain any point with integral coordinates except O, has a volume not exceeding*

* This answer would be of the kind envisaged earlier, since the volume of the ellipsoid is expressible in terms of λ and the determinant of the quadratic form, which is the sole algebraic invariant of the form.

† For brevity, I am using the word *congruent* to mean that the one body can be derived from the other by a translation only.

2^n. In other words, *an ellipsoid with centre O and volume greater than* 2^n *must contain a point with integral coordinates, other than O, in its interior.*

As we have seen, the proof is exceedingly simple. At one point only have we used any special property of the ellipsoid ; namely when we asserted that the mid-point of the centres of two congruent intersecting ellipsoids lies in the common part. Now this holds good for many other bodies, e.g. for parallelepipeds. Minkowski showed that it holds for any body * (in n dimensional space) which is

(i) *symmetrical about O*, that is, if a point belongs to the body, so does the image of that point in O ;

(ii) *convex*, that is, if P and Q are two points of the body, so is their mid-point.

The proof is very simple. Suppose R is such a body, and $R(A)$, $R(B)$ denote congruent bodies centred at A, B. Suppose $R(A)$ and $R(B)$ have a point X in common, and complete the parallelogram $AXBY$. Let X', Y' denote the images of X, Y in A. Since X lies in $R(A)$, so does X' by the symmetry, and consequently Y lies in $R(B)$ by the congruence of $R(A)$ and $R(B)$. Similarly Y lies in $R(A)$. By the convexity, the mid-point of XY lies in both $R(A)$ and $R(B)$, and this point is also the mid-point of AB. The proof can be expressed even more briefly by using an appropriate vector notation.

Thus we have, without further proof, the more general theorem :

Any convex body in n dimensional space, which is symmetrical about O and has volume greater than 2^n, *contains a point with integral coordinates, other than O.*

This is Minkowski's fundamental theorem in the geometry of numbers. It is very effective in solving problems of Diophantine approximation, and was applied by Minkowski to give simple proofs of many theorems in that subject and in algebraic number-theory. In this general form, applying to any symmetrical convex body, it is easily seen to be best possible, in that the number 2^n cannot be replaced by a smaller one. For the cube

$$|x_1| < 1, \ldots, |x_n| < 1$$

is a convex symmetrical body, whose volume is exactly 2^n, and it does not contain a point with integral coordinates other than O.

There is a more general geometrical interpretation than the direct one which we have used up to now. Apply to the variables $x_1, \ldots, x_n$ any linear transformation

$$X_1 = \alpha_{11}x_1 + \ldots + \alpha_{1n}x_n,$$
$$\ldots$$
$$X_n = \alpha_{n1}x_1 + \ldots + \alpha_{nn}x_n,$$

with real coefficients α_{ij} whose determinant Δ is not zero. This represents the general affine transformation from the x space to the X space. The system of points with integral coordinates in x space is transformed into a system of points in X space of a more general kind, which we call a *lattice*. If we denote by $A_1 = (\alpha_{11}, \ldots, \alpha_{n1})$ the point in X space which corresponds to the point $(1, 0, \ldots, 0)$ in x space, and so on, the general point P of the lattice is given by

$$\overrightarrow{OP} = x_1\overrightarrow{OA_1} + \ldots + x_n\overrightarrow{OA_n},$$

where $x_1, \ldots, x_n$ take all integral values. The parallelepiped based on the vectors $OA_1, \ldots, OA_n$ may be called the *fundamental parallelepiped* of the lattice.

* To avoid complications which are irrelevant to the argument, I have not defined the term " body ". It will suffice if we understand it as meaning any set of points which is *open* (*i.e.* every point is an interior point), and possesses a *volume*, in any of the various ways in which " volume " can be defined.

A body of volume V in x space is transformed into a body of volume $V' = V \mid \Delta \mid$ in X space. In particular, the volume of the fundamental parallelepiped is $\mid \Delta \mid$, and we call this the *determinant of the lattice*. The properties of convexity and of symmetry in O are preserved by an affine transformation. We can restate Minkowski's fundamental theorem in X space as follows :

Let L be any lattice of determinant Δ in n dimensional space. Then any convex body, symmetrical about O, whose volume exceeds $2^n\Delta$, contains a point of L other than O.

We can use an affine transformation to simplify any particular body with which we are concerned, at the expense of having to consider a general lattice instead of the special lattice of points with integral coordinates. For example, we can transform any ellipsoid into a sphere, and the problem with which we began becomes the problem of finding conditions under which a sphere will necessarily contain a point of any lattice of given determinant. The answer provided by Minkowski's theorem is, in this particular case, clearly not the best possible. For it was obtained by arguing that a set of non-intersecting spheres cannot more than fill up space, and it is plain that in fact an appreciable proportion of space will be left empty. The problem of finding the best possible constant for a sphere, in place of 2^n, is the same as the problem of the *closest packing* of equal spheres in n dimensions, subject to the condition that the centres of the spheres shall form a lattice. We may call this the closest *regular* packing. A considerable advance was made by Blichfeldt in 1914. He used a physical analogy, involving material spheres with a certain law of density, and counted the total quantity of matter instead of the total volume, thus obtaining* an appreciably better constant than 2^n. The true constant is still not known for general n. In two dimensions, the closest of all packings happens to be regular, but this is probably not the case in higher dimensions.

In many problems of Diophantine approximation, the geometrical interpretation does not give us a convex body. In such cases, we can deduce a crude result by applying Minkowski's theorem to a suitable inscribed convex body. A simple example is that of the region

$$\mid XY \mid < 1$$

in the plane. We ask, how small must be the determinant Δ of a lattice in order that we can be certain that a lattice point other than O will lie in the region? If we apply Minkowski's theorem to the inscribed square

$$\mid X \mid + \mid Y \mid < 2,$$

which has area 8, we deduce that the result certainly holds if $\Delta < 2$. In arithmetical form, the problem is that of the minimum of $\mid XY \mid$, where

$$X = \alpha x + \beta y, \quad Y = \gamma x + \delta y$$

are two linear forms of determinant Δ, and x, y take all integral values. A theorem of Markoff tells us that this minimum does not exceed $\Delta/\sqrt{5}$, and that this result is best possible. Consequently, for the geometrical formulation, the true condition is $\Delta < \sqrt{5}$. The corresponding problem for the body

$$\mid XYZ \mid < 1$$

in three-dimensional space was first solved by me in 1937. The answer is that this body contains a point other than O of every lattice whose determinant Δ satisfies $\Delta < 7$, and that 7 is the true constant. It is hardly possible for me in this lecture to say anything useful about this work; or about the subsequent work of Mordell and Mahler, which has given a new lease of life to the subject, and has unfolded problems of an entirely new kind.

* See a later account by Blichfeldt in *Math. Annalen*, 101 (1929), pp. 605–8.

I will conclude by showing how Minkowski's theorem can be used* to prove a classical result in number-theory: Lagrange's theorem that every positive integer is representable as the sum of four integral squares. This is not a very easy theorem to prove: Euler tried many times between 1730 and 1750 and failed. There is one Lemma which is needed in all the direct elementary proofs of the theorem, and which I shall take for granted. This is that for any odd positive integer m there exist integers a, b satisfying

$$a^2 + b^2 + 1 \equiv 0 \pmod{m}. \tag{1}$$

This is proved by simple considerations relating to quadratic residues when m is a prime p, then by induction on v when $m = p^v$, and finally by combination of these results it follows for general m.

We proceed to prove that m is representable as the sum of four integral squares. We define four linear forms X, Y, Z, W in four variables, x, y, z, w as follows:

$$\begin{aligned} X &= mx + az + bw \\ Y &= my + bz - aw \\ Z &= z \\ W &= w \end{aligned}$$

The determinant of these forms is obviously m^2. Thus if x, y, z, w take all integral values, the corresponding points (X, Y, Z, W) form a lattice in four-dimensional space of determinant $\Delta = m^2$.

In virtue of (1), we have

$$X^2 + Y^2 + Z^2 + W^2 \equiv 0 \pmod{m} \tag{2}$$

for all values of x, y, z, w. If we prove that there is a lattice point other than O for which

$$X^2 + Y^2 + Z^2 + W^2 < 2m, \tag{3}$$

it will follow from (2) that m is representable as $X^2 + Y^2 + Z^2 + W^2$. The inequality (3) represents a sphere of radius $\sqrt{(2m)}$. By Minkowski's theorem, it will suffice to prove that the volume of this sphere is greater than $2^4 \Delta = 2^4 m^2$. Now one easily finds, by integration, that the volume of a four-dimensional sphere of radius r is $\frac{1}{2}\pi^2 r^4$. Hence it suffices if

$$\tfrac{1}{2}\pi^2 (2m)^2 > 2^4 m^2,$$

and this is so, since $\pi^2 > 8$.

This proves Lagrange's theorem for any odd positive integer m, and the result extends at once to even integers, since if †

$$m = X^2 + Y^2 + Z^2 + W^2$$

then

$$2m = (X + Y)^2 + (X - Y)^2 + (Z + W)^2 + (Z - W)^2.$$

I do not say that this is the ideal proof of Lagrange's theorem; far from it. But it is a simple proof, with a minimum of tiresome detail, and it shows how the geometry of numbers can be used to prove purely arithmetical results. I would hazard the conjecture that there are many other applications of this kind awaiting discovery. H. D.

* This proof was given by Hermite in 1853 (*Oeuvres*, I, p. 288). Of course he could not appeal to Minkowski's theorem; in place of it he used his own result on the minimum of a positive definite quadratic form. Professor Forder has drawn my attention to a somewhat similar proof by Grace, *J. London Math. Soc.*, 2 (1927), pp. 3–8.

† In connection with this step, I cannot refrain from referring to Professor Mordell's note in this *Gazette*, 26 (1942), p. 52, on Lewis Carroll's diary. Lewis Carroll found considerable difficulty in proving that if m is a sum of two squares, so is $2m$.

THE INCORPORATION OF STATISTICS INTO A SCHOOL COURSE.*

By B. C. Brookes.

Introduction.

In this account of the incorporation of a new subject into the school curriculum I shall assume that it is agreed that the teaching of statistics in schools is desirable. This aspect of the subject has already been fully discussed at previous meetings of this Association, and in other places. I am one, and I hope there are others present in the audience, who believes not only that it is desirable to teach statistics in schools, but also that far more is gained than is lost by its introduction. In this paper I therefore present the story of one attempt to put into practice what has been preached so long.

The first step to take is to enlist the support of one's headmaster. This is possibly easier if he is not a mathematical scholar. Of the many arguments that can be adduced for introducing statistics the one that is the most welcome to headmasters is likely to be that the new subject can be incorporated into the curriculum without affecting any other subject but mathematics. Having stated that first, the educational reasons can then be given. One good reason that I will mention here, because it has a bearing on the teaching and examination of the subject, is that it gives practice in the analysis and interpretation of numerical data that is lacking in the orthodox school curriculum, that is, it helps to bring school mathematics into closer contact with other school subjects and with the outside world. Fortunately, I can say that I have received all possible encouragement from my own headmaster.

Backed by the support of the headmaster one is then in a stronger position to seek willing victims. I looked round the school—a grammar school of 900 boys—for material on which to make the experiment. The minimum mathematical standard on which a serious course of statistics can be founded is that of a pass in S. C. Elementary mathematics, and so my search was limited to the Sixth Form. From a mathematical point of view there are usually to be found in the Sixth Form four distinct levels :

1. The mathematics specialists.

2. The science or engineering specialists for whom mathematics is an ancillary.

3. The classical, literary or arts scholars who retain sufficient interest in mathematics to be taking it as a subsidiary Higher Certificate subject.

4. The non-academic type of boy who is not intending to take a Higher Certificate or any advanced examination.

I discussed the possibility of teaching statistics with each of these four groups. The enthusiasm of the mathematics and science specialists was tempered by doubts concerning the status of statistics as an examination subject. " How will the learning of statistics affect my prospects of gaining a Higher Certificate and a university entrance scholarship? " they asked. I had to admit that, so far as I know, it is not yet possible to include statistics as an examination subject in entrance scholarships for Cambridge, Oxford, London or anywhere else. The only recruits I am likely to get from these groups at present are therefore from among those who have already satisfactorily arranged for their entrance to a university or profession and have some time to spare. One boy, from the specialist science group, a budding bio-chemist, has recently joined the statistics class.

* A paper read at the General Meeting of the Mathematical Association, April 10, 1947.

The members of the third group are also seeking Higher Certificates and scholarships but mathematics is taken, if at all, only at a subsidiary level. The group contains geographers and economists who appreciate that an introductory course in statistics would be of direct value to their main interests. One form in this group had already spent one year on the usual subsidiary pure mathematics course and were beginning to wonder why. It has always seemed to me that we mathematicians err if we expect candidates in mathematics at the subsidiary level to be satisfied with snippets from the least interesting ends of the course devised for specialists. Usually the candidates taking subsidiary mathematics are intelligent despite the fact that their main interest is non-mathematical! The usual subsidiary course, unless it is well taught by a lively teacher, is a rather drab affair which leaves its victims, if they are of a critical mind, with a stronger prejudice against mathematics and no greater understanding of it than they had when they began the course. The boys of this group, in the second year of their Higher Certificate course, received the suggestion that they should include statistics in their mathematics syllabus with some relief. The Cambridge Local Syndicate authorities readily agreed to substitute the syllabus recommended for subsidiary mathematics by the Cambridge Joint Advisory Committee for the normal H. S. C. subsidiary mathematics. Copies of the pamphlet containing the recommendations of the Advisory Committee have been circulated with a recent number of the *Mathematical Gazette*, so that its contents are probably known to most of those present. However, briefly, the new syllabus contains three main sections, pure mathematics, mechanics and statistics, the mechanics and statistics being alternatives. The old syllabus contained no applied mathematics at all so that no mechanics is being lost by the introduction of statistics.

The selection of this second year literary group as the field for the experimental introduction of statistics had the following advantages :

(i) No other subject had to be squeezed out, and the only loss in mathematics was of some pure mathematics, e.g. some coordinate geometry of the straight line and circle, in which the students were not particularly interested.

(ii) It immediately gave the new subject an examination status. Whatever one's views for or against examinations may be, one cannot ignore the fact that both boys and staff take less seriously those subjects that are not considered suitable for examination. I wanted statistics to be looked on, not merely as an adjunct to a course in Civics, but primarily as a serious branch of applied mathematics.

(iii) It was easier for me as the teacher to start at the most elementary level.

The Statistics Syllabus.

I have already mentioned that the statistics syllabus on which we are working is that suggested by the Cambridge Advisory Committee. So far I have found it to be working very well. It is orthodox in its approach, it is closely correlated with the pure mathematics side of the syllabus and develops very naturally out of it. It is therefore possible to teach the pure mathematics and its application as one undivided subject. Even in this subsidiary syllabus enough of the subject is covered to give the student a sound foundation on which he can himself build a larger structure if he so desires ; if he is content to leave the subject alone after completing this course, enough has been done to enable him to be more intelligently critical of the " figures " that are so freely used in this age of planning.

Syllabuses are, however, open to many interpretations and I think it would be worth while giving you my interpretation of the syllabus on which we are working. The first item is " The tabulation and graphical representation of statistical data illustrated, e.g. by population, trade, growth of plants, examina-

tion marks ". Examination requirements could probably be met by confining the teaching to the histogram, and possibly the frequency polygon, but a wider view of this topic should be taken as it is one of some social importance. In almost every newspaper we find some graphical representation of numbers and I expect my form to bring to our notice any worthy of comment. The questions I hope they ask themselves are : " Is this representation effective ? " " Is it misleading? " For actual exercises the *Monthly Digest of Statistics* published by the Stationery Office provides ample material on trade, collaboration with the biology laboratory will provide data on plants, and if the biology side runs an experimental garden some useful exercises in the statistical planning of experiments and interpretation of results can be obtained. I have found examination marks to be rather dangerous material to work with in a school. All school subjects have their comparative ups and downs but the Masters' Common Room as a whole is unlikely to become enthusiastic about the new subject if it learns that the Sixth Form has confirmatory statistical evidence, that, since the appointment of Mr. X, the standard of History teaching has suffered a marked decline! If you want to use marks, use your own or those published, for example, with the Sandhurst, Woolwich and Cranwell examination papers. The local meteorological data are usually readily available and provide a wealth of exercises.

Frequency distributions are then dealt with quantitatively and the student is introduced to the common measures of location, *i.e.* the arithmetic mean, the mode and the median, and of dispersion, *i.e.* the quartiles, the mean deviation and the standard deviation. Here it is important to expect the pupil to be able to decide for himself under what conditions any one of these measures is preferable to the others, to be able to interpret them and to appreciate the limitations of the information these measures convey. During the War I was once confronted with the argument propounded by a senior officer, that a certain training course must be effective *because* the averages of the marks obtained by any individual member of the course for the first and second, the first, second and third exercises, and so on, tended steadily towards the final average of the ten or twelve exercises ! When I had convinced him that his " proof " was merely a property of any random sequence of numbers I was dismissed with the remark that " Statisticians can prove anything ! " Perhaps my interest in the teaching of statistics is partly explained.

In an expanded and revised edition of the Advisory Committee's syllabus three additional items are included in this section. They are, firstly, the *moving average* which can be introduced as a means of smoothing irregular curves such as those given by economic statistics with a time base. The effect of varying the span of the average when it is used on curves that show a seasonal or other regular periodic variation should be examined.

The second addition is the *index number*, a subject of topical interest, which is considered as a *weighted average*. The school tuck-shop should be able to provide data on which the schoolboy can calculate a cost-of-living index for himself.

The statistic *variance*, of which the standard deviation is the square root, has also been added, presumably because of the development by R. A. Fisher and others of the analysis of variance. Though it is a statistic of increasing interest and importance little more than a descriptive account of its properties is possible at this mathematical level.

So far the mathematical knowledge demanded of the pupil has been merely a modest extension of School Certificate arithmetic and algebra though the use of Σ for " the sum of all terms like ... " may be new. The next section of the syllabus demands a rather higher mathematical attainment as it includes the binomial distribution and the descriptive properties of the normal distribu-

tion. The binomial distribution may be treated as an extension of the usual algebraic work on the binomial theorem and on probability, but the introduction of the normal distribution is more difficult at this level. It can be approached in three ways. All of them should be used.

(i) The graphs of e^{x^2} and e^{-x^2} can be studied as a simple extension of the study of the functions e^x and e^{-x} which are covered in the pure mathematics syllabus.

(ii) The normal distribution can be considered as a limiting form of the binomial distribution for a large exponent. The proof of this is not needed for the examination, but a simplified version can be followed by the pupils; it is instructive in itself and leads to the formula $y = y_0 e^{-x^2/2\sigma}$ which appears less formidable if the function e^{-x^2} is already known to the pupil.

(iii) The normal distribution can be demonstrated practically by experiments with darts, or as the error distribution associated with estimates of a length.

The next section of this syllabus is concerned with the general ideas of simple sampling and applications of the formulae $\sqrt{(pq/n)}$ for testing the significance of a proportion and of $\sigma/\sqrt{n}$ for testing the significance of a mean. This is comparatively straightforward from the mathematical point of view. Statistically it introduces the important concepts of randomness and significance and boys find it interesting as it gives scope for some practical work.

The last item in the Advisory Committee's syllabus has been expanded from "elementary ideas on correlation" to include an elementary graphical treatment of regression. This is a welcome amendment as I think that the idea of regression is more general and also simpler than the idea of correlation. Regression does not imply a causal relationship between the two variables considered, whereas correlation usually does. I believe someone has shown that there is a highly significant correlation between the imports of bananas and the growth of the British Navy. You may smile, but schoolboys—and others—are often attracted by such unexpected relationships and take them more seriously than they should. It is easier therefore for the schoolmaster to stress the point that significant correlation does not imply causal relation if the teaching of regression preceded the teaching of correlation. Though only a graphical treatment is asked for I have in fact shown my class how to *calculate* regression equations because the graphical treatment alone was unconvincing. The class found it difficult to imagine how the *two* regression lines appear, *i.e.* the regression line of x on y, and that of y on x; they were quite satisfied with one line until by calculation they actually found two. The calculation of the equations is straightforward since the function $mx+c$ has already been studied.

It will be seen that this syllabus offers an interesting and instructive field for applied mathematics, with only two serious, but not insoluble problems of presentation for the teacher, the introduction of the normal distribution and the graphical treatment of regression and correlation. It does, however, apply only to Sixth Form mathematics taken at the subsidiary level. Can anything more be reasonably attempted? The Advisory Committee go just one step further and suggest that for Mathematics taken at the Ordinary standard there could be added the standard error of a difference between means in large samples. This addition, together with a more detailed treatment of the subsidiary syllabus, would form an adequate and suitable course for the Ordinary level.

For more advanced work there are several interesting topics that are within the scope of the schoolboy mathematician, and it is not difficult to suggest

additions to the syllabus already discussed. For the mathematical scientist I would suggest the following topics :

1. Continuous and probability distributions for the study of which the simpler methods of integration are required. The ideas of the first and second moments of a distribution could be introduced pointing out the analogies with the centre of gravity and moment of inertia of a plane lamina, with which the pupils will already be familiar. The effect of grouping, and Sheppard's correction for the second moment would be included.

2. A more mathematical treatment of the normal distribution and the Poisson distribution, which, as the distribution of events of rare occurrence, have some application to biology and to physics.

3. A more mathematical treatment of bivariate distributions, with calculation of linear regression lines and correlation coefficients.

4. The Chi- squared distribution and its application as a test of " goodness of fit ".

For the mathematical specialist there could also be added :

5. An introductory treatment of the analysis of variance, and, possibly,

6. Tests of significance for small samples.

From the statistical point of view any student who completed this course would be in a strong position to continue his studies either in the theoretical field, in which there is great scope for the mathematical scholar, or in the field of applied statistics, in which his statistical knowledge can be related to his particular science.

The Relation of Statistics to other Sixth-Form Mathematics.

The drafting of possible syllabuses in statistics is a pleasant and harmless occupation until you try to put them into practice. In order to squeeze statistics into the curriculum something else must be squeezed out. What is this to be? I do not think it is necessary to encroach upon the time allocation outside mathematics.

At the subsidiary level the Advisory Committee offer a syllabus in which Mechanics and Statistics are alternatives. There is no serious loss to mechanics here because most pupils will have done some work in this subject at the School Certificate stage, if not in mathematics then as a branch of physics.

At the Ordinary level the Advisory Committee again have two alternatives, but on a different basis. The statistics may be taken at the expense of the following topics :

Trigonometry—the graphical solution of problems by accurate drawing of sections and projections.

Calculus—the approximate solution of algebraic equations, centres of gravity, moments of inertia and Simpson's rule.

Geometry—the analytical geometry of the conic sections.

Mechanics—rotation of a body about a fixed axis, simple machines, centre of gravity and Bow's notation.

This is a more serious loss though for the science student I think the sacrifice is worth while. Unfortunately, however, the omitted topics are mainly those which are included in any engineering course. The adoption of the alternative including statistics would therefore handicap the engineer in his basic subjects. On the other hand, since the development of quality control there is perhaps no field in which statistical methods are being applied with more enthusiasm than in production engineering. In large schools the engineers can be segregated and taught separately, and in some respects the engineering students' examination problems are different from those of the science specialist. Since,

however, I believe it will shortly be possible for engineering specialists going direct from school to industry to obtain exemption from part of their A.M.I.Mech.E. examinations on the results of Higher Certificate work, the engineers will soon be working largely on the same syllabus as the scientists, and the problem of squeezing in a statistics course at this level remains unsolved.

For those pupils who take mathematics beyond the Ordinary stage the Advisory Committee, so far as I know, have made no detailed suggestions though the matter is under consideration. The problem here is again to decide what should be left out rather than what should be put in. Looking through the recommendations of the Committee, which I hope and assume will be generally adopted, it is difficult to find anything that can obviously be omitted in favour of statistics. Yet I think there is room for the statistics course I have outlined if less time could be spent on mechanics, not by the omission of any part of the mechanics syllabus, but by being satisfied with problems which are exercises in applied mathematics rather than camouflaged exercises in pure mathematics. I think we can also afford to dispense with some of the more artificial problems on which our pupils spend so much time, problems such as those beginning: "A smooth wedge of mass M is placed on a smooth table and carries a small smooth pulley at its summit. Two masses m_1 and m_2 . . .", or of those monuments of human ingenuity and imagination we find in all our standard textbooks under the heading "Harder Friction". During the War I had to solve many mechanical problems such as the calculation of the rate of fire of a newly designed machine-gun, or the time of action of a mechanical fuse, given the working drawings, in which the real problem was always to assign a value to μ, the coefficient of friction. The experience of years of teaching Harder Friction failed to help.

The point I am making here is that, other things being equal, applied mathematics is the better for being *real*. If this statement is acceptable then there can be found plenty of time for teaching statistics as part of the Sixth Form's applied mathematics.

A second argument for including statistics in this way is that almost every mathematical specialist reads physics as another main subject. If he does, then he is taught mechanics experimentally, as I think he should be. I have never been able to understand why examination candidates are allowed to take mechanics in physics and as applied mathematics in mathematics in the same examination as if they were two distinct subjects, and thereby get two sets of marks for the learning of one subject.

A third argument for including statistics as part of the applied mathematics course is that it very greatly extends the field to which the pupil's pure mathematics can be applied.

I know that I could very readily solve the problem of deciding what to omit so that statistics may enter, but it is not so easy to persuade others to agree to my solution. Unfortunately, until there is agreement about what is to be sacrificed neither I nor anyone else can get very far with the teaching of statistics in schools.

Some teaching points.

There are some lesser, but still important, problems that arise in any attempt to incorporate statistics into school teaching. Firstly there is the absence of textbooks. At the subsidiary level this lack of suitable textbooks is not important. It does perhaps make it more difficult for the teacher but it probably enlivens the subject for the taught. For more advanced work there are existing textbooks that are suitable in part, but no book so far as I know has been written specifically for use in schools. I have found five books that are very useful in different ways. They are:

(*a*) Tippett's *Statistics* in the Home University Library—a non-mathematical survey of the subject which gives an excellent idea of the statistician's outlook, and which should be read by every teacher of statistics at least once a year.

(*b*) Levy and Preidel's *Elementary Statistics*, which is useful for its practical outlook and simple exercises. (There is a promised companion volume on Experimental Statistics for which I have waited impatiently for two years.)

(*c*) Yule and Kendall's *Introduction to the Theory of Statistics*—an indispensable reference book for general theory.

(*d*) Bowley's *Elements of Statistics*—written for the economist and useful for reference on subjects like index numbers.

(*e*) Weatherburn's *Mathematical Statistics*—a recent publication. It contains some interesting harder examples and could be left in the hands of the mathematical specialist.

The textbook difficulty is the more seriously felt because so few practising mathematical teachers have had either academic training in statistics or any practical experience of it. In my own school, of ten who teach mathematics I am the only one at present able to teach statistics. Next year I have arranged to hand over my present statistics class to another member of the staff who would like to gain experience of teaching the subject. It is not very helpful to hand over the textbooks to him and leave the matter there : I can of course lend him a copy of my own notes, which may help in this particular case, but something more than this is required. I am hoping that the Ministry of Education or one of the Universities will realise that a Summer School on the teaching of statistics would be popular, and having realised that, do something about it.

The school study of statistics should be as far as possible, experimental and topical. Some apparatus is needed for the experimental work, but it is very simple—so simple indeed that it may give rise to false impressions. One of my most respected literary colleagues came to see me in my mathematics room while my statistics class was experimentally building up a circular normal distribution—with darts ! He made no comment, but looked surprised. Later he came to see me again ; the statistics class was drawing single cards at random from a pile of numbered cards—playing-cards are, of course, the most readily available. His worst fears were confirmed. One of my holiday tasks is to prepare some biassed dice. I hope we shall live it down as the biologists seem to have done.

The remaining teaching points, which it is important continually to stress with beginners, may be summarised :

(*a*) That statistical data must always be examined for origin and reliability before valid conclusions can be drawn.

(*b*) That the secret of accurate computation is neat and careful working together with the systematic application of checks at every stage.

(*c*) That statistics is a comparatively recent development of mathematics and that it lies within the scope of any student to improve or extend its theory or to widen its field of application.

(*d*) That the data required for a statistical analysis consist not only of the numbers and measurements but also a knowledge of all relevant factors.

(*e*) That nothing is ever *proved* by statistics, its conclusions are stated only in terms of probabilities.

The Royal Statistical Society recently appointed a committee to advise on the teaching of statistics but I understand that this committee is at present concentrating on the teaching of the subject in the universities.

Examinations.

What kind of questions are we likely to see in statistics examinations? I do hope that the vitality of the new subject will not be sapped by the setting of stereotyped computations only. What has happened to mechanics could so easily happen to statistics. It is alarming to speculate on what might be the statistical analogues of the mechanics problems on smooth wedges. The attraction of statistics as a teaching subject lies largely in its ability to relate school mathematics to real social, economic and educational problems. If this is to be the reason for teaching statistics it must be reflected from the start in the examination papers. Schoolmasters tend only too readily to teach their subjects to meet the demands, not so much of the syllabus which can always be interpreted liberally or narrowly, but of the examination papers.

I am not training my subsidiary class to become computers, but I hope that they will get a glimpse of the statistician's sensible outlook on "figures" and realise not only the possibilities but also the limitations of statistics. Questions such as :

1. Calculate the s.d. of . . . .
2. What is the coefficient of correlation of . . .
3. Find the median. . . .

can be answered by the computer, and are, I suggest, to be avoided unless some question involving interpretation of the results is included.

From 1948 it will be possible to offer statistics in the Cambridge Local H. C. subsidiary mathematics paper. For the specialists the position is more difficult. Most of my pupils are unable to enter a university unless they win a college entrance or state scholarship. Competition is intense and one therefore hesitates to spend time on a subject however interesting and however great its educational value if it means that one's pupils are handicapped in the competition. If the Cambridge Colleges adopt the recommendations of the Advisory Committee we may soon see one or two statistics questions in the applied mathematics papers. The initiative here, lies, I think, with the universities and colleges and not with the schools, though presumably there might be a response if a request were made by or through this Association.

For the present I shall limit the teaching of statistics to those at the subsidiary level and to those who have passed through the fire and have a term or two at school free of examination worries before going up to the university or commencing their professional training. The rest will be condemned to doing problems in Harder Friction, and I, with growing impatience, to teaching them.

B. C. B.

GLEANINGS FAR AND NEAR.

1544. 1 B.C. and A.D. 1 are both the same year. Of the 365 days, those which fell before the birth of Christ are counted as 1 B.C. and those after as A.D. 1. Thus, 2,000 years have elapsed since 55 B.C.—*News Chronicle*, August 26, 1946. [Per Mr. Ivor Pearcy.]

1545. Bertrand Russell has completed the great work upon which he has been engaged for many years. It is called *A History of Western Philosophy and its Connection with Political and Social Circumstances from the Earliest Times to the Present Day*. The book is formidable in appearance, since it compresses within some 850 pages the thought and studies of a lifetime, but it is not formidable to read, since it is written with that clarity which only mathematicians seem able to achieve.—Review by Harold Nicolson, *Daily Telegraph*, November 22, 1946.

MATHEMATICS AND THE CHILD.

By C. Gattegno.

When teachers consider the problems of mathematics in the school they recognise that they have to make a definite effort to simplify the study of these difficult questions, but, it seems to me, they work in a frame which is imposed by the curriculum and their temptation is to think very little of the child. The problem, however, has to be faced as a whole and the aim of this article is to discuss the teaching of mathematics with special reference to a child-centred point of view.

Syllabuses are produced by adults who tend to think that mathematical knowledge can be divided into parts which can be allocated to children of such-and-such an age. If this view does not correspond to reality, then a new committee decides to replace one question by another seemingly more suitable to the children concerned. *A priori*, one could say that if there are so many changes in the syllabuses, some deeper mistake must lie behind.

What is this mistake? It seems to me to be merely this: the child is ultimately the agent of his own apprehension and his growth of knowledge. Without considering the child as an active unit, as someone to be known before thinking of subjects to be taught to him, we are bound to make mistakes and to undertake continuous amendments of the curriculum, of books and of methods. In fact the art of teaching mathematics is measured today by the ability of the teacher to disclose new methods outside the field of mathematics to convey the truths of mathematics. Is this not an implicit admission of failure? Is there not, therefore, an urgent need to consider the reality as it stands and to think of the teaching of mathematics as the problem concerning mathematics and the child?

In advocating a complete reconsideration of the situation I am quite aware that I shall meet many objections. But one of the qualities of mathematicians (not always shared by other men) is that they know that when there is a mistake somewhere, there is no use hiding it. It will only appear somewhere else in the form of a contradiction. This time the contradiction can be felt when we consider how many individuals dislike mathematics or have a fear of it. This forces mathematicians to write popular books to recapture adult interest.

Historically the development of schools was from schools for grown-ups to schools for all children, and in consequence children were taught without being known, and with the implicit assumption that they were prospective men and only that. Moreover, the needs of industry demanded technicians first of all and it was tacitly agreed that fifteen years of school, spent in acquiring some technical ability, was a good price to pay for a specialist. The curriculum was therefore devised to give at the end of these years the required ability. It was the task of the teacher to get the result, no matter what the method and the effort required: for example, solve fifty exercises on removing brackets, two hundred on simplifying fractions, do six years of drill on multiplication tables, etc. As only the result counted, there was no need to consider the method, there was no room for reform, and still less for consideration of the reality of the child.

But as machines were doing so much work, the need for technicians was both reduced in relation to the school population and increased in quality. More workers of a higher efficiency were required and more had to enter careers where other subjects were more useful. A competitive society needs more lawyers to settle its disputes, more trade travellers to place the goods,

etc. Syllabuses were revised so as to introduce more advanced material an
also to give room for other training. But the methods remained the same
Every now and then a crisis forced the educational authorities to undertake
slight change here and there ; sometimes because a better method was foun
to expound such-and-such a theorem, or even because the opinion of som
authority in a Committee prevailed. But there was not a real feeling fo
reform.

The reform in spirit was imposed from outside. First, it was found that th
child is not a small man but an individuality with its own ways of thinkin
and its own purposes. He, in the end, will be an adult, but this will be late
and, what is very important, he will be an adult in the world of the futur
which nowadays cannot be described with great certainty. Secondly, som
educational pioneers have let the child develop, observed it and given it littl
teaching in the ordinary sense. In these schools sometimes there was n
change of syllabuses but there was an attempt to change the method. It wa
still believed that the aim of education was the transmission of culture and tha
the test for the success of this is the official examination. But it was also
believed that much depends on the way in which the subject is treated and
the child's natural curiosity used. The project method, or the Dalton plan
prepare candidates for the same examinations as traditional methods, bu
convey the material in a new way. These methods were successful. In
America, where everything can be attempted, and finds followers, hundreds
of methods were tried. The result is that now almost everybody who teaches
mathematics wants to appeal to some external help in performing his task.

Today there is an intense interest in methods and an experimental era has
started. Discussion on how to improve the teaching of logarithms is common
today among the teachers. In front of such a situation it seems to me to be
time for a thorough reconsideration of the whole problem of the child and
mathematics.

Today we find people who ask whether it is not a pity to waste so many
years in the learning of mathematics when nothing remains. For us mathema-
ticians, it is impossible to agree with the implication. We feel so strongly
that mathematics is one of the greatest achievements of the human mind, and
enjoy its beauties so much, that we cannot conceive a school curriculum
without mathematics. But the facts are there. Huge efforts at school do not
leave any trace. It is not true that a mathematician is an abler man or a
better man, or even a pleasanter man. His place in society is often that of
a crank. But society cannot progress without mathematicians, and needs
many of them, more and more as the world becomes more mechanised.

We shall find a way out of this dilemma if we consider the problem as a dual
one. There is the problem of the necessity of having mathematicians among
the scientists and there is the problem of securing that mathematics comes into
the school in the natural development of the child. The first problem will be
solved if the blind law, which has provided the world with mathematicians
since the remotest times and without special care, is not opposed. Let people
whose taste for abstractions is great have a chance to meet mathematics at the
proper moment. Therefore the first problem will be solved as a corollary to the
solution of the second problem, which is the really difficult one.

We shall attempt to offer a solution of this problem, at least a theoretical
solution. It is not true that there is only one way of obtaining the truth, and
that this is the method by which geometrical truths are obtained. In fact, if
one studies logic, one finds that there are several possibilities, the axioms of
each being themselves not logical, and it is ultimately a question of taste
which decides the choice. There are several geometries, contradicting each

ier, one of which is taught as true in the schools, the others being simply nored. If we call logic the set of basic " tools " which allow us to obtain ruths ", let us see if what we have found above is not the case for the logic the child. Perhaps it may be decided that our plain adult logic is the only e, and the others may be deliberately ignored. Perhaps all our system of ucation may have been built on that assumption.

Since the development of theories of relativity we have learned to consider fferent observers and not to blame anyone if results are different unless a itable transformation is found to decide the results in any system of reference. ave we undertaken such a study of the different observers in the school case : rselves and the children? Have we thought that our successive views out the universe, about ourselves, about logics, . . . imply the necessity of relativistic view of the problems? A relativistic view is possible. We shall dertake it on the logical lines mentioned above, not as a problem of abstract gic, but of plain philosophy : *logic will be what makes us believe that we have tained the truth.*

If I see my daughter in the street, I recognise her and can go to her, talk ith her about things we know. There is no doubt for me that this girl is my ughter. I have used a judgment which is instantaneous and due to percep- on. I have used *logic of perception*. The same happens if I look at a right- gled triangle or an equilateral triangle. I do not need to think. I recognise ch of them immediately. Obviously, this logic of perception has its limits. f the angle A is 89°, I shall still say that the triangle is right-angled at A. similar fashion, I may make a mistake if I think I see a Mr. Smith whom I nce met in a crowd. The logic of perception exists even though it may not be ecure. If we want to know definitely whether ABC is a right-angled triangle, here is only one way, that of measurement. To agree upon the result is then gic of perception based upon *logic of action* ; similarly for Mr. Smith, we an ask him his name. Logic of action exists in everyone of us. If I tell any- ody that I can walk on my hands from Liverpool to London in ten minutes, e does not need to think, he will state immediately that it is impossible. lthough he has not seen me trying it, his experience lets him judge it im- ossible. He uses logic of action. In the well-known problem which leads to he conclusion that a right angle is equal to an acute angle, when we want to rove that there is a mistake, there is no other way of proving the falsity of the esult than by drawing a figure using circle and ruler. It is not mere reasoning hich is used, but action (the construction). Those who think that geometry the development of abstract notions in the mind, must stop when they neet the previous result, for if no action is used, any following theorem is ndifferently right or wrong.

The Middle Ages are known for the use of *faith* as a tool in the search for ruth. What is in the Bible was true and everything which can be deduced yllogistically from the contents of the Bible was also true. Angles are as real s men. For the Greeks, *beauty* was a criterion of truth. For adults of today, either faith nor beauty is used as a criterion in geometry, but in many cases *motional judgments* are used. There is a *logic of feeling* (or of emotion) which ontains faith and aesthetic feeling as special cases. If we walk in a street nd see a driver of a car deliberately driving over a dog which he could avoid, e throw ourselves on the driver to punish him for his villainous action ithout asking ourselves any question. The response is immediate. Our oral feelings are hurt and do not require any deliberation. The logic used to ind this truth is merely logic of feeling.

Can we find examples in mathematics in which this logic of feeling is in ction? It is a daily experience of the mathematician that an awkward way

usually leads to a mistake and a " fine " method of approach leads to truth. Hundreds of theorems in mathematics are given on assumption. Their proofs are sought for, sometimes for centuries (Fermat's last theorem, the four-colour problem, . . .). Mathematicians believe that these results are right, or that there must be a way of proving them. It is mere faith. The fact that every continuous curve has a tangent, that there is only one infinity, are mere beliefs. Zermelo's axiom still divides mathematicians ; those who do not want to use it call the others " mystics ". Logic of feeling is also used when we approach any problem in which the content gives us joy and happiness.

The logic used in every book of mathematics is usually called the *logic of the obvious* (or, evidence) based on some simple ideas such as : the whole is greater than the part, it is impossible to be and not to be at the same time, the same part of space can not be occupied by two different bodies simultaneously. It is only when we can reduce our problems to obvious ideas that our mind is satisfied. All our books of mathematics proceed by it. We do not need to give any example. But on the contrary we want to show its limits. Zermelo's axiom is an example which shows that what is obvious to one may not be obvious to another. If we have more than one person involved, the obvious must be defined. That is what the positivists did when they decided to use for finding truth in society the *logic of mental convergence* : What we agree upon is true. Has this logic any representatives in the field of mathematics? Obviously all the statistical results are of this type ; they draw their acceptance from this kind of logic. To choose a law to agree with the majority of the results but not with all is a definite new step in the search for truth. Graphical and statistical methods are representatives of the logic of mental convergence and were considered simultaneously in history. Clearly, if some fact is obvious to everyone, it is obvious to the majority, and so logic of the obvious is contained in logic of mental convergence, but not conversely.

We can find other types of logic : *pragmatism*, which is the method of finding truth by the validity of the result ; *logic of invariance*, the relativistic substitute for relative truths.

We have discussed only one point : logic. But obviously it has great importance in relation to the teaching of mathematics. It shows that *if a child goes through these different logics* the syllabuses must be moulded on this evolution. It is unreasonable to use graphs at an early stage, for they are abstractions and not concrete things. They satisfy a man trained in social life and social thinking. Their meaning is acceptable only there.

There must be a world-wide investigation :

(i) of what are the ways used by children to get their truths ;

(ii) of what are the substitutes for the truths we want to hand to the child in the field of mathematics, substitutes which will use the logic present at the stage in which the child is, which will build the next logic as a tool.

By this process we can still keep the teaching of mathematics in the school syllabus, give joy to the child as he does what he understands, not interfere with his growth, build his knowledge in all fields in the same time, obtain his spontaneous adhesion to abstract mathematics when at the age of 14 or 16 his logic of the obvious is his tool of judgment. Normal children, if not disgusted earlier, like algebra and geometry in post-puberty. The knowledge gathered outside the school in their early childhood is the only support for their logic of perception, logic of action, logic of feeling, and they use these elements to support their logic of the obvious. It is because the school does not provide this support that we find boys of 15 who think that a brick can weigh 10 grams or a hundredweight. If the school provides mathematics corresponding to the

existing logic not only will it save the future of mathematics in the schools but also the future of the child in society.

In the space of such a short article it is impossible to discuss thoroughly all the points raised and we ask the reader to consider this article merely as an invitation to consider the problem from this angle. Another article will give some personal attempts to find substitutes for the problems we meet in teaching practice.

C. G.

ANNUAL GENERAL MEETING, 1948.

DIFFICULTIES concerning accommodation for the Annual General Meeting, 1948, have been solved through the co-operation of the Institute of Education, University of London. The meeting will be held in the Malet Street buildings on 1st, 2nd and 3rd January. Cafeteria lunch will be available in the Senate House Large Refectory on Friday, 2nd January; tea will be served in the Common Room of the Institute of Education on Thursday and Friday.

It has also been arranged that the Common Room shall be open to members of the Association for the duration of the meeting, thus providing an additional opportunity for friends to meet together.

The Business Meeting of the Association will commence at 2 p.m. on Thursday, 1st January, and Professor G. B. Jeffery will read his Presidential Address at 2.45 p.m. At 5 p.m. Professor E. H. Neville will give a paper on "What is a number".

On Friday, 2nd January at 10 a.m. and 11 a.m. respectively, Mr. J. T. Combridge will speak on "Replacing Mathematics by Reckoning" and Professor H. W. Turnbull will speak on "Intuitive Geometry". The afternoon session (2 p.m. to 4.15 p.m.) will be devoted to a discussion on "Typography"; various aspects of the subject in relation to mathematics books being considered. At 5 p.m. Professor A. C. Aitken will give a "Demonstration of Arithmetic".

The Saturday morning session (commencing at 10 a.m.) will include a discussion on "Mathematics in School Examinations".

The usual Publishers' Exhibition is being arranged, but the evening Conversazione unfortunately cannot be held this year. The Programme Committee will, however, book seats for a theatre party or other evening function if members will send their requests for this to the Secretary.

F. W. K.

1546. The inhabitants of Philadelphia are an honest, worthy, square, good-looking, well-meaning, regular, uniform, straight-forward, clock-work, clear-headed, one-like-another, salubrious, up-right kind of people, who always go to work methodically, never put the cart before the horse, talk like a book, walk mathematically, never turn but in right angles, think syllogistically, and pun theoretically, according to the general rules of Cicero and Dean Swift, whereas the people of New York—God help them—toss about over hills and dales, through lanes and alleys, and crooked streets, continually mounting and descending, turning and twisting, whisking off at tangents, and left-angle triangles, just like their own queer, odd, topsy-turvy, rantipole city, are the most irregular, crazy-headed, quicksilver, eccentric, whimwhamsical set of mortals that ever were jumbled together in this uneven, villainous, revolving globe, and are the very antipodes to the Philadelphians. —From Washington Irving's *Salmagundi*, quoted in *The World of Washington Irving*, by Van Wyck Brooks (New York and London, 1944), pp. 8–9. [Per Mr. F. Puryer White.]

COMMUTATIVE INVOLUTIONS.

By R. L. Goodstein.

We prove some theorems on commutative involutions in a "real" projective geometry in which cobasal homographic ranges may have 0, 1 or 2 self-corresponding points (and therefore a conic and a general line in its plane have 0, 1 or 2 points of intersection).

Definitions. If x' corresponds to x in a (1, 1) relation $\mathfrak{R}$ we write $x\mathfrak{R}x'$; the relation $\mathfrak{R}$ is involutory if $x\mathfrak{R}x'$ implies $x'\mathfrak{R}x$ for some one pair x, x'. If $\mathfrak{J}$ and $\mathfrak{J}'$ are involutions of points on the same line or conic, and if $x\mathfrak{J}y$, $x\mathfrak{J}'x'$ and $y\mathfrak{J}'y'$ implies $x'\mathfrak{J}y'$ for all pairs x, y of $\mathfrak{J}$ then $\mathfrak{J}$ is said to be *amicable* to $\mathfrak{J}'$.

(We take for granted the dual formulation of amicable involutions of lines passing through a point or touching a conic).

1. If $\mathfrak{J}$ is amicable to $\mathfrak{J}'$ then $\mathfrak{J}'$ is amicable to $\mathfrak{J}$.

For if $x\mathfrak{J}'x'$, $x\mathfrak{J}y$, $x'\mathfrak{J}y'$ and $y\mathfrak{J}'\bar{y}$ then $x'\mathfrak{J}\bar{y}$; but $x'\mathfrak{J}\bar{y}$ and $x'\mathfrak{J}y'$ implies $\bar{y}=y'$, so that $y\mathfrak{J}'y'$ and $\mathfrak{J}'$ is amicable to $\mathfrak{J}$.

1·1. If the involution $\mathfrak{J}$ has two self-corresponding points l, l' then $l\mathfrak{J}'l'$ if, and only if, $\mathfrak{J}, \mathfrak{J}'$ are amicable.

For if $\mathfrak{J}, \mathfrak{J}'$ are amicable and if $x\mathfrak{J}y$, $x\mathfrak{J}'x'$, $y\mathfrak{J}'y'$ then $x'\mathfrak{J}y'$, and so $(ll'xy)=(l'lx'y')=-1$; hence l, l'; x, x'; y, y' are point pairs in the involution $\mathfrak{J}'$.

Conversely, if $l\mathfrak{J}'l'$ and $x\mathfrak{J}y$, $x\mathfrak{J}'x'$, $y\mathfrak{J}'y'$ then $(l'lx'y')=(ll'xy)=-1$, so that $x'\mathfrak{J}y'$ and therefore $\mathfrak{J}, \mathfrak{J}'$ are amicable.

1·2. If there is a *single pair* x, y such that $x\mathfrak{J}y$, $x\mathfrak{J}'x'$, $y\mathfrak{J}'y'$ and $x'\mathfrak{J}y'$ then the involutions $\mathfrak{J}, \mathfrak{J}'$ are amicable.

Let p, q be any point pair of the involution $\mathfrak{J}$, and let $p\mathfrak{J}'p'$, $q\mathfrak{J}'q'$.

We have $(p'x'xy)=(pxx'y')$, from the involution $\mathfrak{J}'$,
$=(qyy'x')$, from $\mathfrak{J}$,
$=(q'y'yx)$, from $\mathfrak{J}'$,

and so $p'\mathfrak{J}q'$, which proves that $\mathfrak{J}$ is amicable to $\mathfrak{J}'$.

1·3. If two involutions are amicable then either is determined by a *single* pair if two pairs of the other are given.

For if $x\mathfrak{J}y$ and $x\mathfrak{J}'x'$, $y\mathfrak{J}'y'$ then $x'\mathfrak{J}y'$ and so $\mathfrak{J}$ is determined by the two pairs x, y; x', y'.

2. $\mathfrak{J}$ and $\mathfrak{J}'$ are involutions on the same line or conic. If $x\mathfrak{J}y$ and $y\mathfrak{J}'y'$ then we write $x\mathfrak{J}\mathfrak{J}'y'$ and $y'\mathfrak{J}'\mathfrak{J}x$.

We prove that $\mathfrak{J}, \mathfrak{J}'$ are amicable if, and only if, $\mathfrak{J}\mathfrak{J}'$ is an involution.

For if $x\mathfrak{J}y$, $y\mathfrak{J}'y'$ and $x\mathfrak{J}'x'$ imply $x'\mathfrak{J}y'$, then $x\mathfrak{J}\mathfrak{J}'y'$ and $y'\mathfrak{J}\mathfrak{J}'x$, so that $\mathfrak{J}\mathfrak{J}'$ is an involution.

Conversely, if $\mathfrak{J}\mathfrak{J}'$ is an involution, and if $x\mathfrak{J}y$, $x\mathfrak{J}'x'$, $y\mathfrak{J}'y'$, then $x\mathfrak{J}\mathfrak{J}'y'$ and so $x\mathfrak{J}'\mathfrak{J}y'$, whence $x'\mathfrak{J}y'$ so that $\mathfrak{J}, \mathfrak{J}'$ are amicable.

2·1. *$\mathfrak{J}$ and $\mathfrak{J}'$ are amicable if, and only if, $\mathfrak{J}$ and $\mathfrak{J}'$ are commutative.*

If $\mathfrak{J}, \mathfrak{J}'$ are amicable and $x\mathfrak{J}y$, $y\mathfrak{J}'y'$, $x\mathfrak{J}'x'$ then $x'\mathfrak{J}y'$ and so $x\mathfrak{J}\mathfrak{J}'y'$ and $x\mathfrak{J}'\mathfrak{J}y'$ for all x, so that $\mathfrak{J}, \mathfrak{J}'$ are commutative; and if $\mathfrak{J}, \mathfrak{J}'$ are commutative and $x\mathfrak{J}y$, $x\mathfrak{J}'x'$, $y\mathfrak{J}'y'$ then $x\mathfrak{J}\mathfrak{J}'y'$, and so $x\mathfrak{J}'\mathfrak{J}y'$, whence $x'\mathfrak{J}y'$ and $\mathfrak{J}$ is amicable to $\mathfrak{J}'$.

3. The points O, O' lie in the plane of a conic Σ. $\mathfrak{J}, \mathfrak{J}'$ are the involutions on Σ determined by families of lines through O, O' respectively.

Then O, O' are conjugate with respect to Σ if, and only if, $\mathfrak{J}, \mathfrak{J}'$ are commutative.

For if $\mathfrak{J}, \mathfrak{J}'$ are commutative and $x\mathfrak{J}y$, $x\mathfrak{J}'x'$, $y\mathfrak{J}'y'$, then $x'\mathfrak{J}y'$, so that $xy, x'y'$ meet at O and xx', yy' at O', which proves that O, O' are conjugate.

Conversely, if O, O' are conjugate and if $x\mathfrak{J}y$, $x\mathfrak{J}'x'$, $y\mathfrak{J}'y'$, then $x'y'$ meets xy at O, conjugate to O', so that $x'\mathfrak{J}y'$.

4. L is the pole of a line l with respect to a conic Σ. A is any point on Σ and m is any line through L. i is the involution on l of pairs of points P, P' conjugate with respect to Σ, and i' is the involution on m of pairs of points Q, Q' conjugate with respect to Σ. $\mathfrak{J}$ is the involution of pairs of lines AP, AP''' and $\mathfrak{J}'$ is the involution of pairs of lines AQ, AQ'.

Then the involutions $\mathfrak{J}, \mathfrak{J}'$ are commutative.

Proof. Let l meet AL and m at B and C respectively.

B', C' are the points on l conjugate to B, C respectively, so that BiB', CiC'. AB', AC' meet m at D, E respectively. Since B and L are conjugate to B', therefore BL is the polar of B', and so ADB' touches Σ at A. Since CLD is the polar of C', therefore AC' is the polar of D. Accordingly C, L and D, E are conjugate pairs on m, so that $Ci'L$ and $Di'E$.

Hence, writing b, b', c, c' for AB, AB', AC, AC', we have proved $b\mathfrak{J}b'$, $b\mathfrak{J}'c$, $b'\mathfrak{J}'c'$ and $c\mathfrak{J}c'$, so that the involutions $\mathfrak{J}, \mathfrak{J}'$ are commutative.

5. The four points A, B, C, D and a line l lie in a plane.

$\mathfrak{J}$ is the involution determined on l by the conics through A, B, C, D.

Σ is one of the conics through A, B, C, D, and $\mathfrak{J}'$ is the involution on l of pairs of points conjugate with respect to Σ.

Then the involutions $\mathfrak{J}, \mathfrak{J}'$ are commutative.

Let l meet AB at P and CD at Q, so that $P\mathfrak{J}Q$. P', Q' are the points on l such that $P\mathfrak{J}'P'$, $Q\mathfrak{J}'Q'$. We prove that $P'\mathfrak{J}Q'$.

If $P'A$, $P'B$ meet Σ again at A', B' then $A'B'$ passes through P, and similarly if $Q'C$, $Q'D$ meet Σ again at C', D' then $C'D'$ passes through Q.

Thus both the pairs of lines AB', $A'B$ and CD', $C'D$ meet at L, the pole of l with respect to Σ.

Suppose further that $P'A, Q'C$ meet at $U, P'B, Q'D$ at V, and AD, BC at W.

Consider the six points A, D, C', C, B, A' on Σ; AD, CB meet at W, DC', BA' at L and $C'C, A'A$ at U, so that, by Pascal's theorem, L, U, W are collinear. Similarly, L, V, W are collinear, and therefore U, V, W are collinear. By Pascal's theorem for the six points P', A, D, Q', C, B it follows that these six points lie on a conic and therefore $P'\mathfrak{J}Q'$, which proves that $\mathfrak{J}, \mathfrak{J}'$ are commutative.

If we say that a set of involutions are *in harmony* if there exists an involution commutative with each involution of the set, then theorem 5 may be expressed by saying that *the involutions, on a line, of conjugate points with respect to conics through four points, are in harmony.*

6. $\mathfrak{J}, \mathfrak{J}_1, \mathfrak{J}_2$ are involutions on a line l, and $\mathfrak{J}$ is commutative with both $\mathfrak{J}_1$ and $\mathfrak{J}_2$. If $A\mathfrak{J}_1P$, $B\mathfrak{J}_1Q$, $A\mathfrak{J}_2X$ and $B\mathfrak{J}_2Y$, then the involution $\mathfrak{J}_3$, determined by the pairs P, Y; Q, X, is also commutative with $\mathfrak{J}$.

We prove first the following Lemma :

6·1. $\mathfrak{J}$ is an involution on a line l in the plane UVL; then there is exactly one conic through U, V, L for which $\mathfrak{J}$ is an involution of conjugate points.

Let VL meet l at T and $T'\mathfrak{J}T$. U' is the harmonic conjugate of T with respect to V, L; UL meets $T'U'$ at E, VE meets UT at D, and $U'E$ meets UD at D', so that $(TD', UD) = -1$. Furthermore, UV meets l at W, $U'C'$ satisfies $(C'W, UV) = -1$ and $W\mathfrak{J}W'$. $C'W'$ meets WL at V' and VL at F, and UF meets WL at C, so that $(WV', CL) = -1$.

Σ is the unique conic through $UVCDL$.

By construction, $U'D'$ is the polar of T and $V'C'$ the polar of W, with respect to Σ, and therefore T, T' and W, W' are conjugate points with respect to Σ. Thus $\mathfrak{J}$ is the involution on l of conjugate points with respect to Σ.

We return now to the proof of theorem 6.

Let U, V be two points in a plane through l, and let AU, BV meet at L. By the foregoing Lemma there is a unique conic Σ through U, V, L such that $\mathfrak{J}$ is an involution of conjugate points with respect to Σ. If PV meets Σ again at M and MU meets L at Q' then, by theorem 5, the involution determined by the pairs A, P; B, Q' is commutative with $\mathfrak{J}$; but $A\mathfrak{J}_1P$ and so, by 1·3, $B\mathfrak{J}_1Q'$, wherefore Q' is the same point as Q. Similarly, if XV meets Σ again at N, then NU meets l at Y. Since Σ is a conic of the pencil through U, V, M, N therefore $\mathfrak{J}$ is commutative with the involution determined by the pairs P, Y; Q, X.

7. Any two collinear involutions are in harmony.

Let $\mathfrak{J}_1, \mathfrak{J}_2$ be any two involutions on a line l, and ABP be a plane through l. There is a unique conic Σ_1 through A, B, P with respect to which $\mathfrak{J}_1$ is an involution of conjugate points. Let C be another point on Σ_1, and let Σ_2 be the unique conic through A, B, C with respect to which $\mathfrak{J}_2$ is an involution of conjugate points. If D is the fourth point of intersection of Σ_1, Σ_2, then the line pairs AB, CD; AC, BD; AD, BC meet l in an involution commutative with both $\mathfrak{J}_1$ and $\mathfrak{J}_2$.

8. The conics through two points, with respect to which a given involution is an involution of conjugate points, meet any line in their plane at point pairs in involution.

Let $\mathfrak{J}$ be the given involution on a line l, and A, B the common points of the family of conics. If m is any line in the plane through l and AB, we have to show that the conics of the family which meet m meet it in point pairs in involution.

Let l meet AB and m at U and V respectively, and let $U\mathfrak{J}U^*$, $V\mathfrak{J}V^*$. X, X_1, X_2 are any points on m, $\Sigma, \Sigma_1, \Sigma_2$ the unique conics through A, B and X, X_1, X_2 respectively, with respect to which $\mathfrak{J}$ is an involution of conjugate points. The conics $\Sigma, \Sigma_1, \Sigma_2$ meet m again at X', X_1', X_2'. l meets AX, BX at P, P' and AX_1, BX_1', AX_2, BX_2' at P_1, P_1', P_2, P_2' respectively, and l meets AX', BX at Q, Q'. If $\mathfrak{E}$ is the involution determined on l by the pencil of conics through A, B, X, X' then $\mathfrak{E}$ and $\mathfrak{J}$ are commutative so that $\mathfrak{E}$ is completely determined by the single pair U, V. Accordingly $\mathfrak{E}$ is also the involution determined on l by the pencil of conics through A, B, X_1, X_1' and the pencil through A, B, X_2, X_2'.

Thus $U\mathfrak{E}V$, $U^*\mathfrak{E}V^*$, $P\mathfrak{E}P'$, $Q\mathfrak{E}Q'$, $P_1\mathfrak{E}P_1'$ and $P_2\mathfrak{E}P_2'$. Hence

$$(XX'X_1X_2) = A(XX'X_1X_2) = (PQP_1P_2) = (P'Q'P_1'P_2')$$
$$= B(X'XX_1'X_2') = (X'XX_1'X_2')$$

and therefore the point pairs X, X'; X_1, X_1'; X_2, X_2' are in involution.

R. L. G.

1547. TOP SPINNING UPRIGHT is kept in that position by the combination of two forces—(1) gravitation, which pulls it downwards to the earth; and (2) centrifugal forces, which pulls it horizontally in all directions at once. As soon as the speed becomes too slow to resist the gravitation, the vertical position is lost; but while the speed is great enough to resist the gravitation, the pace is about "a mile a minute". The motions of the planets are controlled by precisely the same laws.—*Pears' Cyclopaedia*, General Knowledge section. [Per Mr. A. E. Williams.]

THE CROSS AND THE FOUNDATIONS OF EUCLIDEAN GEOMETRY.

By H. G. Forder.

1. The main aim of this note is to give a new foundation of Euclidean geometry based on the notions of "*cross*" and "*signed angle*". The reckoning with crosses, or directed angles, advocated by D. K. Picken * has many merits: the rules are few, general, and simple, and one proof covers a number, often large, of special cases. I will first sketch the method, assuming Euclidean geometry already known.

In brief, a cross relates two *lines* and not, like an angle, two *rays*; it has a sign or direction of turn, and multiples of two right angles are ignored. If the cross between the *lines* h, k be denoted by $\measuredangle hk$, then we write $\measuredangle kh = -\measuredangle hk$. If the lines h, k, l pass through the same point, then, in whatever order they occur as we move round the point, we have

$$\measuredangle hk + kl = hl.$$

If h, k are the lines PH, PK we shall write $\measuredangle HPK$ instead of $\measuredangle hk$, and then since *lines* and not *rays* are involved,

$$\measuredangle HPK = H'PK', \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

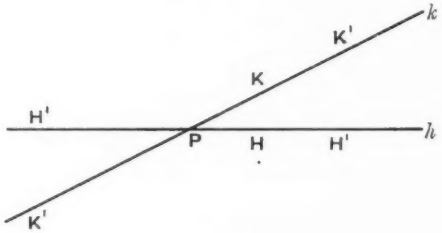


Fig. 1.

where H', K' are *any* points on the *lines* PH, PK. The rules above then give, for all positions of the points,

$$\measuredangle HPK = -KPH \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\measuredangle HPK + KPL = HPL. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

The theorem of the sum of the angles of a triangle becomes

$$\measuredangle BCA + CAB + ABC = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

If AB is parallel to CD, then

$$\measuredangle CAB = ACD. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

If AB is perpendicular to BC, then

$$\measuredangle CBA = ABC. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

If $AB = AC$, then $\measuredangle ABC = BCA$ $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$

Very important is the following: if A, B, C, D be on a circle, however

* *Math. Gazette*, Dec. 1922, p. 188; *Proc. London Math. Soc.*, 23 (1924), p. 45, and later papers in the *Gazette*.

arranged, then

$$\measuredangle ACB = ADB,\ BAD = BCD, \text{ and so on.} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

(The pair of outer letters stays put). This replaces both " angles in same segment " and " opposite angles of a cyclic quadrilateral ".

The reader to whom crosses are not familiar may work through the simple proof * of the " *Pivot theorem* " :

If A', B', C' be *anywhere* on the *lines* BC, CA, AB respectively (whether on the *sides* of the triangle or the sides *produced*) then the circles $B'C'A$, $C'A'B$, $A'B'C$ meet.

For suppose the first two meet in O, then, using (1), (5)

$$\measuredangle CB'O = AB'O = AC'O = BC'O = BA'O = CA'O.$$

Hence by the converse to (8), O lies on the circle $A'B'C$.

Another simple example is a slight extension of the Simson line property :

If P be on the circle ABC and X, Y, Z lie on lines BC, CA, AB respectively and

$$\measuredangle PXB = PYC = PZA,$$

then X, Y, Z are collinear.

For $$\measuredangle PXY = PCY,$$

(since $PXC = PXB = PYC$, and so X, C, P, Y are concyclic),

$$\measuredangle PCY = PCA = PBA = PBZ,$$

$$\measuredangle PBZ = PXZ,$$

(since $PXB = PZA = PZB$, and so X, B, P, Z are concyclic).

Hence $\measuredangle PXY = PXZ$, and the converse of (1) shows the theorem.

2. It has never been explicitly stated, so far as I know, that a large pocket of Euclidean geometry can be concisely formulated in abstract fashion in terms of crosses.

Let us assume that any three points A, B, C, distinct or not, in that order, constitute a cross ABC. This is a real number,† modulo 1. This is now to be regarded as undefined, but the reader can use the work in § 1 as an *interpretation*. Taking numbers modulo 1 corresponds to ignoring two right angles. We assume five axioms :

Ax. I. For all positions of the points

$$APB + BPC = APC.$$

Deductions. Let $A = B$, then $APA + APC = APC$.

Hence $$APA = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Let $A = C$, then $APB + BPA = APA = 0$.

Hence $$APB = -BPA. \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

We can now write I in symmetric form :

$$APB + BPC + CPA = 0.$$

Ax. II. $$BCA + CAB + ABC = 0.$$

* Many examples will be found in Forder, *Higher Course Geometry* (1931).

† It is not really necessary to assume this. Only a very few properties of real numbers are used, and these could be stated in terms of our symbols.

Deduction. $ABC + BCD + CDA + DAB = 0.$ .....(3)

For (by I), the left-hand side is equal to

$$ABC + (BCA + ACD) + CDA + DAB$$
$$= (ABC + BCA) + (ACD + CDA) + DAB$$
(by II, (2)) $$= BAC + CAD + DAB$$
(by I) $$= BAD + DAB = 0.$$

Ax. III. If A, B, C are distinct and $ABC = 0$, then $ACB = 0$.

Deductions. By (2), if $ABC = 0$, then $CBA = 0$. This and III show that then all arrangements of A, B, C, supposed distinct, give zero.

If A, B, C be distinct and $ABC = 0$, we say that A, B, C are "*collinear*", and C is on AB, and so on.

If A, B, C be collinear, and D be any point, then

$$ABD = CBD. \quad \text{.....(4)}$$

For, (by I), $ABD = ABC + CBD = CBD$.

Ax. IV. If A, B, C, D be distinct, and $ABC = ADC$, then $ACB = ADB$.

Deductions. If $ABC = ADC$, then $BAC = BDC$.

For the hypothesis gives $CBA = CDA$. Hence, by IV, $CAB = CDB$.

To justify the definition A, B, C, D are "*concyclic*" if and only if $ABC = ADC$, an attention to the arrangement of the letters (the outer pairs) shows that it is only necessary to show further that :

if $ABC = ADC$, then $BAD = BCD$.

This follows from (3) since the hypothesis and conclusion may be written as $ABC + CDA = 0$ and $BCD + DAB = 0$.

To complete the theory of collinearity it is necessary and sufficient to prove that :

if A, B, C, D be distinct and B, D lie on AC, then A, C lie on BD.

Now, by IV, if $ABC = ADC = 0$, then $ACB = ADB$. Hence by III, $ADB = 0$, and so A is on BD.

We can also treat "parallel" lines. By (3)

if $ABC + BCD = 0$, then $CDA + DAB = 0$.

Hence if $ABC = DCB$, then $CDA = BAD$. Then if X be any point on AB, Y any point on CD, we have

$$XBC = ABC = DCB = YCB,$$

which justifies the definition :

AB and CD are "*parallel*" if $ABC = DCB$.

We can show that such lines, if distinct, do not meet. For, if they meet in E, then

$$ABC = EBC, \quad DCB = ECB.$$

Hence $$EBC = ECB, \quad EBC + BCE = 0.$$

Hence, by II, $CEB = 0$, and so by III, $EBC = 0$, $ABC = 0$ and C is on AB.

Practically all the theorems which can be treated by means of crosses, and their number is large indeed, can now be deduced, but as we sometimes need auxiliary lines, we assume also :

Ax. V. If P, A, B be given distinct non-collinear points, there is a point Q on AB such that PQA or PQB is equal to a given cross.*

* Q might by accident coincide with A.

3. How far will this take us in constructing Euclidean geometry? We can show Pappus' theorem in the parallel case by an easy translation of a known proof.*

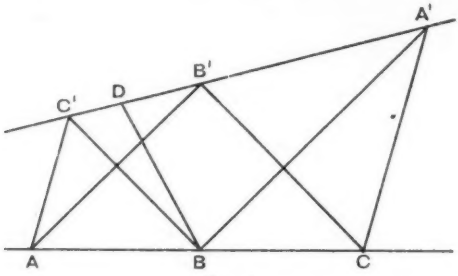


FIG. 2.

If A, B, C are collinear, and A', B', C' collinear, and $A'C$ be parallel to AC' and $B'C$ to BC', then so is $A'B$ to AB'.

That is, if $AC'A' = CA'C'$ and $CB'C' = BC'B'$, then $AB'A' = BA'B'$.

For take D on $A'B'$ so that $A'BC = A'DC$.

Then (i) $BA'D = BCD$, (ii) $AC'D = CA'D$, (iii) $CB'D = BC'D$.

Now interchange B, C in (i): $CA'D = CBD$.

Hence by (ii) $AC'D = CA'D = CBD = ABD$.

Interchange A, B: $BC'D = BAD$.

By (iii) $CB'D = BC'D = BAD = CAD$.

Interchange A, C. $AB'D = ACD = BCD = BA'D$, by (i).

Thus $AB'A' = BA'B'$.

Now we may assume as a known fact,† that from this can be deduced Desargues' theorem on perspective triangles: if the sides of one triangle be parallel to those of another, the joins of corresponding vertices are either concurrent or parallel.

The theory of proportion now follows, as far as this concerns ratios of intervals in the same line; and this is as far as we can go conveniently and naturally.

4. Can we introduce congruence? For intervals along parallel lines we can say $AB = CD$, the lines AB, CD being parallel, if AC, BD are parallel lines. From the case of Desargues' theorem when the joins of corresponding vertices are parallel we can show that this relation is transitive, and extend it to intervals on the same line. The intervals are to be regarded as having a sense or direction.

To define the congruence $AB = AC$ where A, B, C are non-collinear, we naturally take

$$AB = AC, \text{ if and only if } ABC = ACB.$$

But then we must assume at least some cases of the transitive law:

$$\text{if } AB = AC \text{ and } AC = AD \text{ then } AB = AD.$$

* Forder, *Foundations of Euclidean Geometry*, (1927), p. 154.

† Forder, preceding reference, p. 214, or the last edition (1930) of Hilbert's *Grundlagen*, p. 111.

Finally, if AB, XY are not parallel lines we say $AB = XY$ if, and only if, there is a point C with $AC = AB$ and AC equal and parallel to XY. For consistency we must prove that if XZ is equal and parallel to AB, then $XY = XZ$. This follows from Desargues' theorem.

We also need the congruence which concerns two sides and the included angle of a triangle, in the case when both the "included" angles have a common vertex (the rotation case).

Most of the usual theory can now be deduced.

6. But obviously not all. For we have as yet no order-relations. Connected with this omission is the difficulty which Picken encountered in proving that if $OA = OB = OC = OD$ then A, B, C, D are concyclic in the sense of the above definition.

We can hardly expect to introduce order by means of crosses which do not distinguish between an angle and its supplement. Let us make a fresh start, and now use "directed angle". Such an angle is a relation between two *rays, with a sense of rotation*, the kind introduced at the beginning of geometry in schools, and then forthwith discarded.

Proceeding abstractly, if A, B, C be any points, then ABC is a number, mod. 2; (not, as before, mod. 1).

We may keep $Ax.$ I unchanged. Replace $Ax.$ II by

$Ax.$ II′. $$BCA + CAB + ABC = 1.$$

Then (3) follows as before. Replace $Ax.$ III by

$Ax.$ III′. If $ABC = 1$ then A, B, C are distinct and $ACB = 0$.

Now say B is "*between*" A and C if, and only if, $ABC = 1$. The usual properties of this relation between three points now follow, noting that $1 = -1$, since our numbers are taken modulo 2.

To complete the theory of intermediacy along a line, we only need: if A, B, C, X are collinear and B is between A, C then either B is between A, X, or B is between X, C, and this follows from

$$ABC = ABX + XBC.$$

As for order in the plane, and the theorems about the separation of a plane by a line on it, these come most easily if we assume:

BCA, CAB, ABC all lie between 0 and 1 or all between 1 and 2.

Now that we are using angles and not crosses, we must define, when A, B, C are non-collinear:

$$AB = AC \text{ if, and only if, } ABC = -ACB.$$

An adaptation of a common proof then shows that if $OA = OB = OC = OD$ then $DAB + BCD = ABC + CDA$. Hence by (3) both sides $= 1$ or 0, and hence

$$ABC = \pm ADC.$$

APPENDIX

1. We get a symbolism curiously like the above if we interpret ABC to be the area of triangle ABC, with a sign depending on how we move round the triangle. Thus $ABC = -CBA$. With this meaning of ABC we have at once

$$APB + BPC + CPA = ABC \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

for any points A, B, C, P.

2. Let us now proceed abstractly. If P, Q, R be any points, PQR now means a real number, positive, negative or zero, not now taken modulo 1 or 2.

We assume (1) as our basic formula.

Circulating A, B, C this gives at once

$$ABC = BCA = CAB, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

from the properties of addition of real numbers. In (1) take $A = B = C$. Thence $3\,APA = AAA$, which taking $P = A$ gives at once $AAA = 0$. This kind of deduction cannot be used when numbers are taken modulo 1 or 2. Hence $APA = 0$ for all P; and so, by (2), $PAA = 0$, $AAP = 0$. In (1) put $C = B$ and use the results just obtained.

Thence $$APB + BPA = 0,$$

which enables us to complete (2) as follows:

$$ABC = BCA = CAB = -BAC = -ACB = -CBA, \quad \ldots\ldots\ldots\ldots (3)$$

and to rewrite (1) as

$$ABC - BCD + CDA - DAB = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

3. Guided by geometrical properties, let us say that "*C is on the line AB*" when $ABC = 0$. Then A is on the line BC and B on the line AC. But to get the usual collinearity properties we must further assume:

if two terms in (4) vanish, so does one (and hence both) of the remaining terms.

Hence if C, D be on the line AB then A, B are on the line CD. From (1) we find: if P is on the line AB then, for any C,

$$APC + PBC = ABC.$$

4. We will define "*AB is parallel to CD*" to mean

$$ABC = ABD. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Then by (4), $CDA = CDB$ and so CD is parallel to AB. We want this to be a relation between the *lines* AB and CD, and not merely between the point pairs A, B and C, D. For this to be so, we must show that if X, Y be on AB and (5) holds then $XYC = XYD$. This can be done if we assume as a final assumption that

if X be on AB and (5) holds then $AXC = AXD$.

Then if X_1, Y_1 be on CD and (5) holds, we have also $ABX_1 = ABY_1$. It follows from this that AB, CD cannot meet. For if O were on both lines, then $ABX = ABO = 0$.

5. We can now construct "affine" geometry. As Hilbert pointed out, Pappus' theorem can be shown from the theory of areas. Thus with the same figure as before, if $AB, A'B'$ meet in O, then since AC' is parallel to $A'C$, and BC' to $B'C$, we have $CAA' = CC'A'$, and hence from the last sentence of § 3

$$OAA' = OCA' + CAA' = OCA' + CC'A' = C'A'C + A'OC = C'OC = OCC'.$$

Similarly $OBB' = OCC'$. Hence $OAA' = OBB'$, and from this, easily, AB' and $A'B$ are parallel.

As we said before, from this follows Desargues' theorem on perspective triangles, in the case when corresponding sides are parallel, and all the theory of proportion; but a curious observation may be made. So far, we have assumed no existence theorems except those trivial ones, left unstated, which give the symbols a meaning. But to deduce Desargues' theorem from Pappus' theorem we have to draw auxiliary lines and so need genuine existence assump-

tions enabling us to join two points, and to draw parallels through given points to given lines, and to find the cut of two non-parallel lines. From these we can also show the Euclidean parallel axiom, and the transitivity of the relation of parallelism.

Thus the above can be regarded as an economical basis for affine geometry. It cannot be extended to metric geometry without enlarging the scope of ideas involved.

6. Finally for the reader who likes simple formal exercises, I suggest the following, where $ABC, \ldots$ are real numbers.

(1) If $ABC + BCD + CDA + DAB = 0$

for all points A, B, C, D, distinct or not, and if

$$AAD = ADA = DAA,$$

then $$ABC = 0 \text{ for all } A, B, C.$$

(2) If $$ABC - BCD + CDA - DAB = 0$$

for all points A, B, C, D, distinct or not, and if

$$AAD = ADA = DAA, \text{ and further } AAA = 0,$$

then

$$ABC = BCA = CAB = -ACB = -CBA = -BAC.$$

H. G. F.

1548. "We expect prices to rise, perhaps 20% in the next few years. In other words your £ of to-day may buy 16s. worth of goods and services in 1952."—*National News-Letter*, Supplement 550, January 23, 1947. [Per Mr. C. B. Gordan.]

1549. We must not let ourselves slip into the error of assuming that institutional relations and personal relations cover, between them, the whole field of human experience. There are certain human experiences that do not take the form of relations of any kind with other human beings—for example, such experiences as those of mystical religion or aesthetic perception (*Anschauung*) or mathematical apprehension (which Plato considered to be the only perfect pleasure in life). It may be noted, however, that mystics and artists and mathematicians are seldom so divinely or so bestially unsocial as to be content to keep their experiences to themselves. As a rule, they feel an impulse to communicate their individual experience, and this impulse is apt to be strong in proportion to the intensity of the experience which is the object of it. . . . Thus, while it is true that there are certain human experiences which do not take the form of relations with other human beings, it cannot be said that these non-social experiences have nothing to do with social life. On the contrary, these non-social experiences are socially creative just because they are individually intense; and it is their potent social effect that gives them part—though, of course, only part—of the importance which they are universally recognised as possessing.—Arnold J. Toynbee, *A Study of History*, I (1945), p. 454, foot-note 3.

1550. Now my father detested reality. In the man-made world of mathematical symbols he was at home. Triangles behaved as one expected them to, and whatever a woman might do, the angle at the hypotenuse remained constant.—Kenneth Hare, *No Quarrel with Fate* (Sampson Low), p. 29. [Per Mr. G. L. Parsons.]

MATHEMATICAL NOTES.

1977. *Higher mathematics and numerical computation.*

In a recent article in *Nature* (Vol. 157, p. 571, May 4, 1946) entitled "Mathematics and the Civil Service", John Todd and D. H. Sadler expressed the view that training in numerical methods should be given to mathematical students at the post-graduate level. The chief reason given was that their experience had shown that a knowledge of the more advanced parts of mathematics often becomes a powerful tool in the hands of the computer. The object of this note is to draw attention to a further example which arose recently in connection with my research work, providing additional evidence for their views.

In the course of a certain investigation, it was necessary to tabulate the function given by the doubly infinite series

$$\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{\sin(2r-1)\pi\alpha/a\,.\,\sin(2s-1)\pi\beta/b}{(2r-1)(2s-1)\{(2r-1)^2b^2+(2s-1)^2a^2\}} \quad \ldots\ldots\ldots\ldots\ldots(1)$$

where $a=160$, $b=60$, for a range of values of α and β given by

$$\alpha=5{\cdot}0,\ 6{\cdot}0,\ 7{\cdot}0,\ 8{\cdot}0,\ 9{\cdot}0,\ 10{\cdot}0,$$
$$\beta=0{\cdot}6,\ 0{\cdot}7,\ 0{\cdot}8,\ 0{\cdot}9,\ 1{\cdot}0,\ \ 1{\cdot}1.$$

Consider the direct computation from (1) for the values $\alpha=5{\cdot}0$, $\beta=0{\cdot}6$. After considerable numerical computation it was shown that the sum of the first hundred terms of the doubly infinite series (corresponding to $1\leqslant r\leqslant 10$, $1\leqslant s\leqslant 10$) evaluated to $3{\cdot}528\times10^{-7}$. However, the series converged so slowly that even with this amount of laborious calculation little accuracy could be expected.

It may be shown, however, by a method the details of which I hope to publish elsewhere, that one of the double series may be summed, and the expression reduces to

$$\frac{\pi}{4a^2}\sum_{s=1}^{\infty}\frac{\sin(2s-1)\pi\beta/b}{(2s-1)^3}\times\{1-\cosh(2s-1)\pi\alpha/b+\coth(2s-1)\pi a/b\,.\,\sinh(2s-1)\pi\alpha/b-\sinh(2s-1)\pi\alpha/b\,.\,\operatorname{cosech}(2s-1)\pi a/b\} \quad \ldots(2)$$

Moreover, for the given values of a and b, correct to at least five places of decimals we have $\coth(2s-1)\pi a/b$ equal to unity. With this simplification, the expression may be shown to reduce to

$$\frac{\pi^4\beta}{32a^2b}(1-\beta/b)-\frac{\pi}{4a^2}\sum_{s=1}^{\infty}\frac{\sin(2s-1)\pi\beta/b\,.\,\exp(1-2s)\pi\alpha/b}{(2s-1)^3}. \quad \ldots\ldots(3)$$

I have been unable to complete the summation contained in (3), but the series converges sufficiently rapidly for a good approximation to be obtained from consideration of the first ten terms. For the particular case $\alpha=5{\cdot}0$, $\beta=0{\cdot}6$, expression (3) gives a value $3{\cdot}712\times10^{-7}$ which differs appreciably from the value $3{\cdot}528\times10^{-7}$ found by the much more laborious method of direct computation. As it was required to evaluate the expression for the range of values of α and β, the total time saved was considerable.

This example shows that a knowledge of the more advanced parts of mathematics may be extremely useful in problems of numerical computation, and confirms the views previously expressed that training in numerical methods suitable for candidates proposing to make a career in the Scientific Civil Service should be given at the post-graduate level. T. J. WILLMORE.

1978. *On Note* 1825 (XXIX, May 1945).

Note 1825 suggests the theorem :

If $R(x)$ is rational and has no poles in the finite interval $a \leqslant x \leqslant b$, then

$$\int_a^b R(x)\,dx = \Sigma \text{res.} \left\{ R(z) \log \frac{z-b}{z-a} \right\}$$

where the right-hand side denotes the sum of all the residues of the function $R(z) \log \{(z-b)/(z-a)\}$.

Proof. Put $L(z) = \log \{(z-b)/(z-a)\}$, and let δ denote the "dumbbell" contour with end-points at $z=a$ and $z=b$. Then $L(z)$ is analytic, and $R(z)L(z)$ is analytic except for poles, at all points in the z-plane external to δ, including the point at infinity. Consequently, if C is a contour that surrounds δ and all the finite poles of $R(z)$, we have by Cauchy's theorem

$$\int_C = \int_\delta + 2\pi i \{\text{sum of finite residues of } R(z)L(z)\},$$

the integrand in both integrals being $R(z)L(z)$; and hence

$$\int_\delta = -2\pi i (\text{sum of all residues, including that at infinity}).$$

The theorem follows, since, as may be easily shown,

$$\int_\delta = \int_\delta R(z) \log \frac{z-b}{z-a}\,dz = -2\pi i \int_a^b R(x)\,dx.$$

Example. Put $R(z) = z^n$, where n is an integer, positive, negative or zero. Then, near $z=0$,

$$z^n \log \{(z-b)/(z-a)\} = z^n \{\log (b/a) - (b-a)z - \tfrac{1}{2}(b^2 - a^2)z^2 - \dots\};$$

and near $z=\infty$,

$$z^n \log \{(z-b)/(z-a)\} = z^n \{-(b-a)/z - (b^2-a^2)/2z^2 - \dots\}.$$

If $n \geqslant 0$, the only pole of $z^n L(z)$ is $z=\infty$, where the residue is $(b^{n+1} - a^{n+1})/(n+1)$. If $n \leqslant -1$, the only pole is $z=0$, where the residue is $\log (b/a)$ if $n=-1$, or $-(b^{-m+1} - a^{-m+1})/(m-1)$ if $n = -m < -1$. The theorem now gives the well-known results.

Note 1. If we put $\zeta = (z-a)/(b-z)$, the theorem reduces to one given in Watson's Cambridge Tract, *Complex Integration* (§ 29), but the form given here seems simpler.

Note 2. The theorem is given by Harkness and Morley, *Theory of Functions*, p. 197, under the severe restrictions that the poles of $R(z)$ should be simple and that $R(z) = O(1/z^2)$ at $z=\infty$, thus excluding, for instance, the case of $R(z) = z^n$ for every value of n. F. BOWMAN.

1979. *An area and a volume.*

1. *The area of a triangle.*

With rectangular axes in the plane of the triangle, the equations to the sides can be written

$$a_r x + b_r y + c_r = 0, \qquad r = 1, 2, 3.$$

If

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \qquad D = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

where A_1 is the co-factor of a_1 in Δ, and so on, then it is well known and easy to prove by determinant multiplication that $D = \Delta^2$; the coordinates of the vertices of the triangle can be found in terms of the co-factors and hence we have the known result for the area

$$S = \Delta^2 / 2C_1 C_2 C_3.$$

2. *The volume of a tetrahedron.*

Using rectangular axes, the four faces have equations

$$a_r x + b_r y + c_r z + d_r = 0, \quad (r = 1, 2, 3, 4). \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Let Δ be the fourth-order determinant of the coefficients and D the determinant of the corresponding co-factors, as in 1. Then by multiplication, $D\Delta = \Delta^4$ and hence if $\Delta \neq 0$, $D = \Delta^3$.

If P is the point common to the first three planes of the set (1), then

$$x_P : y_P : z_P : 1 = A_4 : B_4 : C_4 : D_4,$$

with similar results for the remaining vertices Q, R, S of the tetrahedron. Thus the volume V of the tetrahedron is given numerically by

$$V = \tfrac{1}{6}\begin{vmatrix} x_P & y_P & z_P & 1 \\ x_Q & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ x_R & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ x_S & \ldots\ldots & \ldots\ldots & \ldots\ldots \end{vmatrix} = \begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ A_4 & \ldots\ldots & \ldots\ldots & \ldots\ldots \end{vmatrix} \div 6D_1 D_2 D_3 D_4$$

$$= \Delta^3 / 6 D_1 D_2 D_3 D_4.$$

(*a*) If $\Delta = 0$ but none of the co-factors D vanishes, the planes have a common point.

(*b*) If $\Delta \neq 0$ but one only of the D's vanishes, the figure is in general an infinite triangular prism.

(*c*) If three of the D's vanish, the fourth must vanish and Δ vanishes also.

Other particular results may be obtained by placing restrictions on the A's, B's, C's. J. P. McCARTHY.

1980. *On Note* 1862 (XXX, February 1946).

The equation of the circle through the feet of the normals from a point (α, β) to the parabola $y^2 = 4ax$ can be found shortly and straightforwardly by parametric methods.

The normal at $(at^2, 2at)$ is

$$y + tx = 2at + at^3,$$

which passes through (α, β) if

$$at^3 + (2a - \alpha)t - \beta = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

If the roots of this equation are t_1, t_2, t_3, let the circle through the corresponding points on the parabola be

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

The circle cuts the parabola where

$$a^2t^4 + 4a^2t^2 + 2gat^2 + 4fat + c = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

If the remaining root is t_4, then $t_1 + t_2 + t_3 + t_4 = 0$. But, from (1), $t_1 + t_2 + t_3 = 0$. Hence $t_4 = 0$. Thus the circle passes through the vertex.

Putting $c = 0$ and removing the corresponding factor t from (2), the resulting cubic

$$at^3 + (4a + 2g)t + 4f = 0$$

must be identical with (1). Comparing coefficients we have

$$2g = -(2a + \alpha), \quad 4f = -\beta.$$

Thus the equation of the circle is

$$x^2 + y^2 - (2a + \alpha)x - \tfrac{1}{2}\beta y = 0. \qquad \text{P. C. UNWIN.}$$

1981. *On Note* 1862.

An easier "trick" for showing that the feet of the normals from (α, β) to $y^2 = 4ax$ lie on

$$2x^2 + y^2 - 2\alpha x - \beta y = 0$$

is as follows.

At any point of $y^2 = 4ax$,

$$2 \log y = \log(4a) + \log x.$$

Thus

$$\frac{2}{y}\frac{dy}{dx} = \frac{1}{x}.$$

The gradient of the tangent is thus $y/2x$ and hence that of the normal is $-2x/y$. But if the normal at (x, y) goes through (α, β), the gradient of the line is $(y - \beta)/(x - \alpha)$. Hence

$$(y - \beta)/(x - \alpha) = -2x/y,$$

giving the required result.

LAWRENCE CRAWFORD.

1982. *The missing clue.*

The following puzzle is one which I have found suitable for 12-year-olds.

1	2	3	4
5		6	
7			
8		9	

John is a boy living in Birmingham who was born on the 8 ac. day of the 7 dn. month in the year 1 ac.; his age in days on July 1st, 1946, is 3 dn.; on December 1st, 1951, he will still be 1 dn. years old.

5 ac. $= \sqrt{(1 \text{ ac.})}$, 6 ac. $= \frac{1}{2}y$, 7 ac. $= 11y$, 2 dn. $= x^2$, 4 dn. $= 7z$.

How old is John's sister?

Other problems of about the same standard would be of interest to many teachers.

M. A. PORTER.

1983. *On the rapid computation of squares.*

One of the most practical uses of the factors of $a^2 - b^2$ which I have come across is for the rapid calculation of the squares of numbers. Re-arranging the standard formula, we have

$$a^2 = (a + b)(a - b) + b^2.$$

To calculate 87^2, for example, choose $b = 3$; then

$$87^2 = (90 \times 84) + 9 = 7569.$$

In practice all the working can be done mentally, the answer being written down from the right-hand side, starting with 9 (the value of b^2) followed by the result of the simple multiplication 9×84. The cases when $b = 4$ or 5 require a 1 or 2 to be carried, but this is readily done.

With a little practice the squares of all numbers up to 99 can be written down using this method, in less time than they can be extracted from tables, and I have found it very useful in statistical work (for example, in correlation coefficients) in which squares have to be computed in large numbers. The method is equally applicable to numbers larger than 99, of course, but then it calls for considerable mental agility. Educational statistical computations, however, will rarely involve squares of numbers greater than 100.

M. H. FREEMAN.

1984. *A Special Case of the General Theorem of Maxima and Minima of n Restricted Variables.*

The following special case of the general theorem, the proof of which can be understood by boys in the sixth form, lends itself to some useful applications.

If $f''(x) > 0 (< 0)$ in the interval (a, b), if $x_1, x_2, \ldots, x_n$ all lie in the interval (a, b) and if $x_1 + x_2 + \ldots + x_n = c$, where c is constant, then $f(x_1) + f(x_2) + \ldots + f(x_n)$ is a minimum (maximum) when $x_1 = x_2 = \ldots = x_n = \frac{c}{n}$.

By the mean value theorem or by drawing the curve it is easily seen that if $f''(x)$ is of constant sign in an interval and x_1, x_2, are in the interval, then $f(x_1) + f(x_2) \gtrless 2f\{\frac{1}{2}(x_1 + x_2)\}$ according as $f''(x) \lessgtr 0$ in the interval.

Now if the x_r are not all equal, replace two that are not, each by their arithmetic mean. Then Σx_r is unaltered, but $\Sigma f(x_r)$ is decreased (increased). Hence $\Sigma f(x_r)$ is least (greatest) when $x_1 = x_2 = \ldots = x_n = c/n$.

These are some of the applications of the above.

1. *A polygon having a given number of sides and circumscribing a circle has least area when it is regular.*

For let the polygon have n sides and let the sides subtend angles $2x_r$ at the centre. Then $\Sigma x_r = \pi$ and all the x_r lie in the interval $(0, \frac{1}{2}\pi)$.

The area of the polygon is $R^2 \Sigma \tan x_r$, where R is the radius of the circle, and $\frac{d^2}{dx^2}(\tan x) = 2 \sec^2 x \tan x > 0$ in $(0, \frac{1}{2}\pi)$. Hence $\Sigma \tan x_r$ is a minimum when $x_1 = x_2 = \ldots = x_n = \pi/2n$, *i.e.* the area is least when the polygon is regular.

2. *A polygon having a given number of sides and inscribed in a circle has greatest area when it is regular.*

For let the polygon have n sides and let the sides subtend angles x_r at the centre. Then $\Sigma x_r = 2\pi$ and all the x_r lie in the interval $(0, \pi)$.

The area of the polygon is $\frac{1}{2}R^2 \Sigma \sin x_r$, where R is the radius of the circle, and $\frac{d^2}{dx^2}(\sin x) = -\sin x < 0$ in $(0, \pi)$. Hence $\Sigma \sin x_r$ is a maximum when $x_1 = x_2 = \ldots = x_n = \pi/n$, *i.e.* the area is greatest when the polygon is regular.

3. *The geometric mean of a set of positive numbers, not all equal, is less than their arithmetic mean.*

Let the numbers be $x_1, x_2, \ldots x_n$, all in the interval $(0, \infty)$, where $\Sigma x_r = c$.

Then $$\Sigma \log x_r < n \log (c/n),$$

since $$\frac{d^2}{dx^2}(\log x) = -\frac{1}{x^2} < 0 \text{ in } (0, \infty).$$

Hence $x_1 x_2 \ldots x_r < (c/n)^n$ or $\sqrt[n]{(x_1 x_2 \ldots x_n)} < (x_1 + x_2 + \ldots + x_n)/n$.

4. *If $0 < p < 1$, then $\sum_{r=1}^{\infty} \frac{1}{r^p}$ is divergent.*

The numbers 1, 2, ... n are all in the interval $(0, \infty)$ and their sum is $\frac{1}{2}n(n+1)$. Hence

$$\sum_{r=1}^{n} \frac{1}{r^p} > \frac{2^p n}{(n+1)^p}, \text{ since } \frac{d^2}{dx^2}\left(\frac{1}{x^p}\right) = \frac{p(p+1)}{x^{p+2}} > 0 \text{ in } (0, \infty).$$

Let $n \to \infty$. Then as $p < 1$, $n/(n+1)^p \to \infty$. Hence $\sum_{r=1}^{\infty} \frac{1}{r^p}$ is divergent.

L. R. B. EHRENBERG.

1985. *Tests of divisibility.*

The first test for divisibility by 7 given by Mr. Clarke (Note 1919) may be generalised and then adapted for numbers greater than 10. Here is the test for divisibility by 19.

Multiply the leading digit by 9 and *subtract* the next digit; multiply the result by 9 and *add* the next digit, and so on, subtracting or adding at any stage any multiple of 19. If the final result is a multiple of 19, so is the original number.

Example. 3086322 is a multiple of 19, for:

$3\times9=27$	$27-0=27$	$27-19=8$
$8\times9=72$	$72+8=80$	$80-4(19)=4$
$4\times9=36$	$36-6=30$	$30-19=11$
$11\times9=99$	$99+3=102$	$102-5(19)=7$
$7\times9=63$	$63-2=61$	$61-3(19)=4$
$4\times9=36$	$36+2=38$	$38=2\times19.$

The rules for 17 and 13 are similar, writing 7 or 3 in place of 9 and 17 or 13 in place of 19.

The proof is very similar to that given by Mr. Clarke; it depends on $x+y$ being a factor of x^n+y^n if n is odd and of x^n-y^n if n is even.

C. DUDLEY LANGFORD.

1986. *Tests for divisibility.*

As I am in the habit of factorising numbers when I cannot get to sleep, I can add to Mr. Parameswaran's * list of tests of divisibility.

Since $100a+b\equiv0 \pmod{399}$ implies that $a+4b\equiv0 \pmod{399}$, we can test for divisibility by 3, 7 and 19 simultaneously by multiplying the last two digits of a number by 4, adding them to the next two, and repeating the process till we are left with only two digits. Thus to test 178,431 we proceed:

1784	19
124	32
1908	51

51 is divisible by 3, but not by 7 nor by 19. The same is true of 178,431. Again since $1000a+b\equiv0 \pmod{2001}$ implies that $2b-a\equiv a-2b\equiv0 \pmod{2001}$, and since $2001=3\times23\times29$, we test for divisibility by 23 and 29 by finding the difference between twice the last three digits and the number of thousands, repeating the process if necessary. Thus 178,431 is divisible by 23 or 29 if and only if $431-2\times178=75$ is so divisible, which is clearly not the case. Other similar methods can be derived from the facts that $102=2\times3\times17$, $201=3\times67$, $301=7\times43$, $1003=17\times59$, $3999=3\times31\times43$, and so on.

Perhaps tests of divisibility in quadratic fields are even more hypnagogic. Thus if $a+b$ is divisible by 2, $a+ib$ is divisible by $1+i$; if $2a+b$ is divisible by 5, $a+ib$ is divisible by $2+i$; if $3a+2b$ is divisible by 13, $a+ib$ is divisible by $3+2i$, and so on. In fact, if you can't achieve sleep by factorising in $K(1)$, you may hope to do so in $K(i)$ or $K(\omega)$.

J. B. S. HALDANE.

1987. *Three integrals.*

The following result of contour integration is to be found in most books on this topic (*e.g.* Goursat, *Cours d'Analyse*, II, p. 109):

$$\int_0^\infty \cos(x^2)\,dx=\int_0^\infty \sin(x^2)\,dx=\frac{1}{\sqrt{2}}\int_0^\infty e^{-x^2}\,dx. \quad \text{.....(I)}$$

It is established by integrating $\exp(-z^2)$ round the contour of Fig. 1. The value of the third integral is then assumed; some books (though not Goursat) state that it can not be found by contour methods.

* Note 1920, *Math. Gazette*, XXX, 164 (1946).

The following integration leads directly to the values of the first two integrals in (I) and hence enables all three to be established by contour integration. *

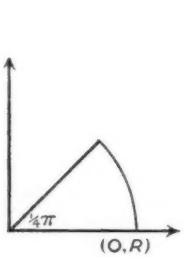


FIG. 1.

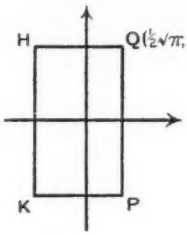


FIG. 2.

Consider $$\int \frac{e^{iz^2}}{\sin z\sqrt{\pi}}\,dz$$

taken round the contour of Fig. 2. The sum of the contributions of PQ and HK is

$$2\int_{-R}^{R} e^{i(\frac{1}{4}\pi - y^2)} \,.\, i\,dy.$$

On QH, $|\sin z\sqrt{\pi}| > \sinh R\sqrt{\pi}$ and $|e^{iz^2}| = e^{-2Rx}$; thus the contribution of QH is in modulus less than

$$\int_{-i\sqrt{\pi}}^{i\sqrt{\pi}} \frac{e^{-2Rx}}{\sinh R\sqrt{\pi}}\,dx = \frac{1}{R}.$$

There is a simple pole of residue $1/\sqrt{\pi}$ at the origin and hence

$$\int_0^\infty e^{i(\frac{1}{4}\pi - y^2)}\,dy = \tfrac{1}{2}\sqrt{\pi};$$

thus,

$$\int_0^\infty \sin(\tfrac{1}{4}\pi - y^2)\,dy = 0, \qquad \int_0^\infty \cos(\tfrac{1}{4}\pi - y^2)\,dy = \tfrac{1}{2}\sqrt{\pi}.$$

Hence expanding and solving for the first two integrals in (I) their common value is found to be $\frac{1}{2}\sqrt{(\frac{1}{2}\pi)}$, and that of the third is then $\frac{1}{2}\sqrt{\pi}$.

J. H. CADWELL.

1988. *A real matrix representing an equilateral polygon in general Euclidean space.*

The obvious generalisation of the equilateral triangle for n-dimensional space is the set of $n+1$ distinct points,

$$P_j \equiv (x_{j,1}, x_{j,2}, \ldots x_{j,n}) \qquad (j = 1, 2, \ldots n+1)$$

such that the distance of any two is the same. This polygon is represented by the matrix of its coordinates $X_n \equiv (x_{j,k})$ of $n+1$ rows and n columns.

* Since writing this note I am indebted to the Editor for drawing my attention to Mordell's evaluation of

$$\int_{-\infty}^{\infty} \frac{e^{at^2+bt}}{e^{ct}+d}\,dt$$

by contour methods, in *Quarterly Journal*, 48 (1920), pp. 329–342. It is effected for real and complex values of the coefficients.

Adding the condition that its centroid should be at the origin, we construct an equilateral polygon by the following iterated process :

If $P_1, P_2, \ldots P_m$ are the vertices of an equilateral m-gon in the $(m-1)$-space, with its centroid at the origin, we place them with their $m-1$ coordinates unchanged into the m-space, adding to each the *same* m-th coordinate $x_{j,m} = \alpha_m > 0 \quad (j = 1, 2, \ldots m)$. Then we add the vertex $P_{m+1} \equiv (0, 0, \ldots 0, -m\alpha_m)$ ensuring that the centroid remains at the origin.

We now determine α_m so that the distances P_1P_2 and P_mP_{m+1} are equal. Taking in the one-space $P_1 = 1/\sqrt{2}$, $P_2 = -1/\sqrt{2}$, *i.e.* $P_1P_2 = \sqrt{2}$, and since by our process in the m-space

$$P_m \equiv (0, 0, \ldots -(m-1)\alpha_{m-1}, \alpha_m)$$

we obtain the recurring relation for $\alpha_m > 0$, $m > 1$,

$$(m+1)^2\alpha^2_m + (m-1)^2\alpha^2_{m-1} = 2,$$

with $\alpha_0 = 0$, $\alpha_1 = 1/\sqrt{2}$.

The solution of this equation is

$$\alpha_m = \{m(m+1)\}^{-\frac{1}{2}}, \qquad m = 1, 2, \ldots, n.$$

The matrix in n-space, when written out, is

$$X_n \equiv \begin{vmatrix} (1\,.\,2)^{-\frac{1}{2}}, & (2\,.\,3)^{-\frac{1}{2}}, & \ldots & \{n(n+1)\}^{-\frac{1}{2}} \\ -(1\,.\,2)^{-\frac{1}{2}}, & (2\,.\,3)^{-\frac{1}{2}}, & \ldots & \{n(n+1)\}^{-\frac{1}{2}} \\ 0 \quad , & -2(2\,.\,3)^{-\frac{1}{2}}, & \ldots & \{n(n+1)\}^{-\frac{1}{2}} \\ 0 \quad , & 0 \quad , & \ldots & \{n(n+1)\}^{-\frac{1}{2}} \\ \ldots \quad , & \ldots \quad , & \ldots & \ldots \\ 0 \quad , & 0 \quad , & \ldots & -n\{n(n+1)\}^{-\frac{1}{2}} \end{vmatrix}$$

It can be verified that the length of every row-vector OP_j is $\{n/(n+1)\}^{\frac{1}{2}}$ and that the distance between any two points represented by row-vectors is $\sqrt{2}$.

In the Euclidean space of infinite dimensions (Hilbert space) we obtain in similar way the infinite matrix

$$X \equiv \begin{vmatrix} (1\,.\,2)^{-\frac{1}{2}}, & (2\,.\,3)^{-\frac{1}{2}}, & (3\,.\,4)^{-\frac{1}{2}}, & \ldots \\ -(1\,.\,2)^{-\frac{1}{2}}, & (2\,.\,3)^{-\frac{1}{2}}, & (3\,.\,4)^{-\frac{1}{2}}, & \ldots \\ 0 \quad , & -2(2\,.\,3)^{-\frac{1}{2}}, & (3\,.\,4)^{-\frac{1}{2}}, & \ldots \\ 0 \quad , & 0 \quad , & -3(3\,.\,4)^{-\frac{1}{2}}, & \ldots \\ \ldots \quad , & \ldots \quad , & \ldots \quad , & \ldots \end{vmatrix}.$$

Here the length of every row-vector is 1, and the distance between any two points represented by row-vectors is $\sqrt{2}$. The polygon is therefore inscribed in the hypersphere $\Sigma_k |x_{j,k}|^2 = 1$. Since the infinite unit matrix I has the same properties, *i.e.* it represents the same polygon rotated about the origin, X is an orthogonal matrix, and so is its transpose X'.

Hence the points represented by the columns of X must form an equilateral polygon.

It is easy to verify the following properties of X : $XX' = I$, $X'X = I$, the length of every column vector is 1, and the distance between any two points represented by column vectors is $\sqrt{2}$. P. VERMES.

1989. *Diameter-closed sets.*

The diameter of a closed set l is defined as the maximum distance between any pair of its points. If we add to l another closed set, we may or may not increase its diameter.

We define a closed set l to be *diameter-closed in a space* $\mathfrak{S}$, if l belongs to $\mathfrak{S}$, and the addition of any point X of $\mathfrak{S}$ not belonging to l increases the diameter of l.

In the *Euclidean plane* the circle is an example for a diameter-closed set. Other obvious examples are regular arc-polygons with an odd number of sides,

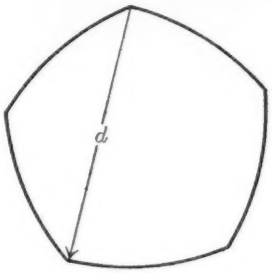


FIG. 1.

the arcs joining consecutive corners of the regular polygon, and having centres at opposite corners. The diagram shows a 5-sided regular arc-polygon.

It can easily be verified, that the diameter of a regular arc-polygon is less than the diameter of the circumscribed circle.

In the Euclidean plane a diameter-closed set is a convex region. For if A and B are two distinct points of l and X any point in AB, $AX < AB$ and $BX < AB$, hence X can be added to l without increasing its diameter. Also there are other points of l than the closed interval AB, since a point not in AB but near its centre would not increase the diameter of the line-segment AB.

It may be of interest to note that some plane figures can be made diameter-closed in several ways, others in one way only.

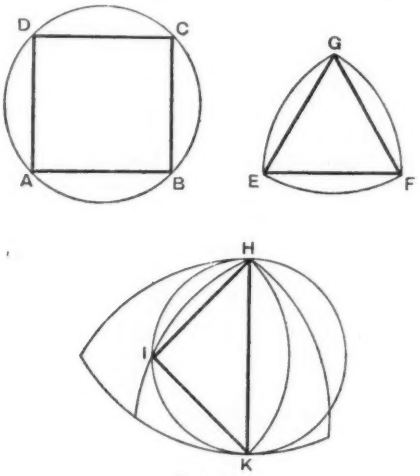


FIG. 2.

Fig. 2 shows plane figures made diameter-closed, namely :

the square $ABCD$ in one way by a circle,
the triangle EFG in one way by an arc-triangle,
the triangle HIK in several ways by arc-triangles and in one way by a circle.
For this triangle there are also an infinity of ways by other odd-sided regular arc-polygons.

In *Euclidean three-dimensional space* there are two obvious examples :
(*a*) the sphere,
(*b*) the regular spherical tetrahedron, bounded by four equilateral spherical triangles, the centres of the spheres being at the corners of a regular tetrahedron, the spheres passing through the other corners.

The other four regular polyhedra have diametrically opposite corners, thus they are of the even-sided polygon type, and therefore not suited for construction of diameter-closed solids. Our theorem on convexity readily extends to this space.

In general *n-dimensional Euclidean space* and in the *Euclidean space of infinite dimensions* (Hilbert-space) the hypersphere is an obvious example for diameter-closed sets. Another diameter-closed set can be constructed using the generalised equilateral polygon described in a recent paper.* In n-dimensional space the polygon has $n+1$ corners, all equidistant from each other (as the regular tetrahedron in 3 dimensions). The diameter-closed set is then the common part of all spheres having centres at the corners of the polygon, and passing through all the other corners. In Hilbert-space, the construction would be similar and the set is not empty, since it contains a neighbourhood of the origin. The theorem on convexity holds in all cases. The results can be extended to complete metric spaces. Whether there are other examples for diameter-closed sets, is open to investigation.

P. Vermes.

1990. *A sensational descent.*

Wanting recently an unmanufactured example on which to test a cascade process for solving the Diophantine equation $ax - by = 1$, I took the leading digits from e and π and applied the method to the equation

$$271828x - 314159y = 1.$$

The clue to the process is to write the equation in the form

$$(272.10^3 - 172)x - (314.10^3 + 159)y = 1.$$

Since $13.157 - 15.136 = 1$, the substitution

$$x = 157y_1 - 15x_1, \qquad y = 136y_1 - 13x_1$$

implies

$$272x - 314y = 2x_1$$

and must effect a substantial reduction in the coefficients ; we find in fact that the equation in the new variables is

$$6647x_1 - 48628y_1 = 1.$$

Also, since the determinant of the transformation is unity, integral solutions of one equation correspond to integral solutions of the other.

The process can be repeated, but before applying it again we write the second equation in the form

$$6647(x_1 - 7y_1) - 2099y_1 = 1.$$

The approximate ratio of the coefficients is now that of 665 to 210, that is, of 19 to 6, and since $1.19 - 3.6 = 1$, the second substitution is

* P. Vermes. "A real matrix representing an equilateral polygon in general Euclidean space", *Mathematical Gazette*, preceding note.

$$x_1 - 7y_1 = x_2 - 6y_2, \quad y_1 = 3x_2 - 19y_2.$$

The result of this substitution is startling: the equation becomes

$$350x_2 - y_2 = 1,$$

and we have at once a solution $x_2 = 0$, $y_2 = -1$, from which we pass back through $y_1 = 19$, $x_1 = 139$ to

$$x = 898, \quad y = 777.$$ E. H. N.

1991. *Acute or obtuse?*

In the latest number of the *Mathematics Student* (XIII, pp. 104-107), Prof. Krishnaswami Ayyangar establishes by means of cross-ratios the theorem that, if $S(x, y) = 0$ represents a pair of lines not at right angles, the point P is in the acute or the obtuse angle between them according as $(a + b)S(x, y)$ is negative or positive, the axes being assumed rectangular. As an excuse for writing out an alternative proof, let me suppose the angle between the axes to have an arbitrary value ω.

The hyperbola through P which has the lines $S(x, y) = 0$ for asymptotes is $S(x, y) = S(x_P, y_P)$, and the orthoptic circle of this hyperbola, if it exists, is

$$X^2 - 2XY\cos\omega + Y^2 - (a - 2h\cos\omega + b)\{S(x, y) - S(x_P, y_P)\} = 0,$$

where $X = ax + hy + g, \quad Y = hx + by + f.$

The terms of the second degree in this equation are $(h^2 - ab)(x^2 + 2xy\cos\omega + y^2)$, and since the lines are assumed to be real, $h^2 - ab$ is positive and the expression on the left of the equation is positive for large values of x and y. It follows that the circle exists if this expression is negative at the centre, and since $S(x, y)$ is a line-pair, this is a point where X, Y, and S are all zero. Hence

The point P is in the acute or the obtuse angle between the lines $S(x, y) = 0$ according as $(a - 2h\cos\omega + b)S(x_P, y_P)$ is negative or positive.

A similar argument answers directly another question, which Prof. Krishnaswami Ayyangar solves as a corollary: If $S = 0$ is a hyperbola, is it acute or obtuse? In this case the circle is

$$X^2 - 2XY\cos\omega + Y^2 - (a - 2h\cos\omega + b)S = 0.$$

At the centre, $X = 0$, $Y = 0$, and S reduces to $gx + fy + c$; eliminating x, y between

$$ax + hy + g = 0, \quad hx + by + f = 0, \quad gx + fy + (c - S_c) = 0$$

we have $(ab - h^2)S_c = \Delta$, and since $ab - h^2$ by hypothesis is negative, S_c has the opposite sign to Δ, and the hyperbola is acute or obtuse according as $(a - 2h\cos\omega + b)\Delta$ is negative or positive.

Doubtless these proofs are old, but the Note in the *Student* is evidence that they are not so familiar that republication is superfluous. E. H. N.

1992. *Application of an identity in elementary algebra.*

The identity is that if $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$. To apply this to finding the equation whose roots are the cubes of the roots of

$$px^3 + qx^2 + rx + s = 0,$$

we substitute $y = x^3$ or $x = y^{1/3}$, and we have

$$py + qy^{2/3} + ry^{1/3} + s = 0$$

or $$(py + s) + qy^{2/3} + ry^{1/3} = 0,$$

whence $$(py + s)^3 + q^3y^2 + r^3y = 3(py + s)qry.$$

T. H. W. H

1993. *The equation, in areal coordinates, to the circle circumscribed to the triangle of reference.*

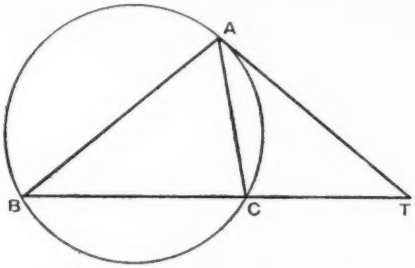
The equation must be of the form

$$fyz + gzx + hxy = 0,$$

since it is satisfied by (1, 0, 0), (0, 1, 0), (0, 0, 1). The tangent at A is

$$gz + hy = 0$$

and so $BT/TC = -h/g$.



But by a well-known theorem in elementary geometry,

$$BT/CT = c^2/b^2,$$

and so

$$h/g = c^2/b^2,$$

giving, by symmetry,

$$f/a^2 = g/b^2 = h/c^2 ;$$

whence the equation

$$a^2yz + b^2zx + c^2xy = 0.$$

From this the equation to any circle may be deduced in the usual way.

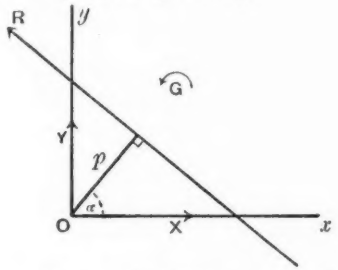
I have not seen this method in any of the standard books; it seems to rest on an easier geometrical premiss than any proof I have come across.

T. H. W. H.

1994. *The equation to the line of action of the resultant of a system of co-planar forces.*

Suppose that the system reduces to forces X, Y acting along Ox, Oy together with a couple of moment G, and that, with the usual notation, the equation to the line of action of the resultant, R, is

$$x \cos \alpha + y \sin \alpha = p.$$



Then

$$X = -R \sin \alpha, \quad Y = R \cos \alpha, \quad G = pR,$$

and so

$$xY/R - yX/R = G/R$$

or

$$xY - yX = G.$$

I find that boys appreciate this method more readily than the method usually given.

T. H. W. H.

1995. *On the definition of logarithms.*

Not being a mathematics teacher I enter this discussion * with hesitation; but here is an approach which I have found very useful at times. It seems to have the great advantage that there is very little to unlearn or redefine later.

We take two points O and U at unit distance apart, and through U we draw an equiangular spiral with centre O, *i.e.* a plane curve such that at each point P on it the tangent makes a fixed angle ϕ with the radius OP (Fig. 1).

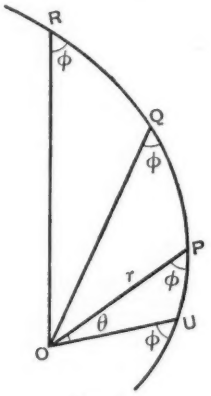


FIG. 1.

Such a curve, it may be pointed out, is a generalisation of the familiar straight line ($\phi = 0$) and circle ($\phi = \frac{1}{2}\pi$); it is the curve described by a moth approaching a bright light. It may also be drawn by a mechanical device such as that

FIG. 2.

shown in Fig. 2, where a wheel has its axis fixed at an angle $\frac{1}{2}\pi - \phi$ with a rod which is constrained to pass through the point O, but is otherwise free.

If P is any point on the spiral, we now define the logarithm of the distance $OP = r$ to be simply the angle $\angle UOP = \theta$, so that

$$\theta = \log r, \qquad r = \text{antilog } \theta.$$

It is now clear why we have many systems of logarithms, since the definition depends on the angle ϕ of the spiral and on the unit of angle for measuring θ. Two fairly obvious alternatives suggest themselves: we may take ϕ to have its most natural value of $\pi/4$ and measure θ in theoretical units, radians. This gives the Napierian system, $\theta = \ln r$, $r = \exp \theta$. Alternatively we take a spiral such that antilog $1 = 10$, and get common logs. A third possibility, instructive for more advanced students, is to take the spiral to be the straight line joining U to one of the circular points.

* *See* R. J. A. Barnard, *Gazette*, XXX (1946), p. 105; R. C. Lyness (same page); P. Gant (same vol., p. 277).

The properties of the logarithm follow immediately from the definition. We see that $\log r$ is 0 when $r=1$, negative when $r<1$, and positive when $r>1$. Furthermore if we take two other radii of the spiral, OQ, OR, such that $\angle QOR = \angle UOP$, then it is evident "intuitively" that the sectors UOP, QOR are of the same shape and differ only in size (Fig. 1). Hence

$$OP/OU = OR/OQ \quad \text{or} \quad OR = OP \,.\, OQ.$$

and $$\angle UOR = \angle UOP + \angle UOQ \quad \text{or} \quad \log OR = \log OP + \log OQ.$$

These two equations constitute the addition theorem for logs. As for differentiation, we know by standard methods that $rd\theta/dr = \tan\phi$ (where θ is in radians), and in particular for Napierian logs, $\phi = \pi/4$, $d\theta/dr = 1/r$.

From the spiral the student will see how we may get rough numerical values for logarithms, but he may well wonder how we can calculate them to many decimal places. Let us therefore calculate (for example) common log 2. Firstly, we see from inspection that $0 < \log 2 < 1$, which gives us the value to the nearest integer.

We now calculate 2^{10}, thus: $2^2 = 4$; $2^4 = (2^2)^2 = 1{\cdot}6 \,.\, 10$; $2^8 = 2{\cdot}56 \,.\, 10^2$; $2^{10} = 2^2 \,.\, 2^8 = 1{\cdot}024 \,.\, 10^3$. We see immediately that $\log 2^{10} = 10 \log 2$ lies between 3 and 4, so that $\log 2$ lies between ·3 and ·4, thus giving its value to one decimal place. Going on in this way to 2^{100} we get $\log 2$ to two places, and so on: the method is quick and practical.

Consider now the equation $\log x^n = n \log x$. This is easily proved when n is a positive integer (supposing x is positive). But, whatever n may be, there will be a number whose logarithm is $n \log x$: it is tempting therefore to call this number x^n. If we do so the usual properties, $x^{-n} = 1/x^n$, $x^n \,.\, x^m = x^{n+m}$, etc., follow immediately, and there is no need to consider separately the cases of n negative integral, n fractional, n irrational, etc. And the definition gives an immediate and direct method of calculating x^n for given n and (positive) x. One important corollary is that antilog $n = (\text{antilog } 1)^n$.

If we draw any two radii OQ, OR, to the spiral, such that $\angle QOR = \theta$, then antilog θ is the ratio $OR : OQ$. This makes less surprising the analogies between antilogs (and especially $\exp\theta$) and trigonometric ratios, such as the irrationality of their values, their addition theorems, series expansions, and formulas for their derivatives.

Finally, this approach gives a simple explanation of a slide rule. Because of its properties as explained above one might think that an equiangular spiral was a useful curve to possess. But in practice it is more convenient to take a circle and mark at each point (θ) on its circumference the value of the corresponding radius (r) of the spiral, which will be chosen so that its radius increases 10 times per revolution. This will in effect give one scale of a circular slide rule, the most convenient type: but we may readily straighten it out if we so desire.

CEDRIC A. B. SMITH.

1996. *The logarithm function.*

Mr. R. C. Lyness' letter (XXX, No. 289, p. 105, May, 1946) stimulates me to indicate the method whereby, in investigating the function

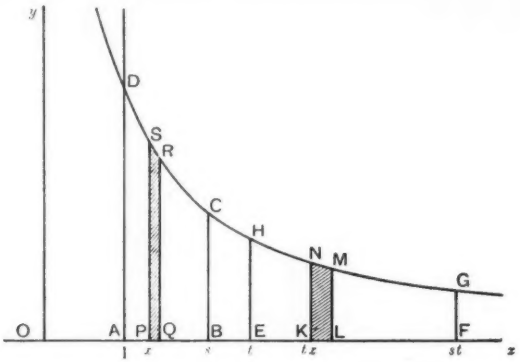
$$\text{hyp}\,(t) = \int_1^t dx/x,$$

I demonstrate the theorem

$$\text{hyp}\,(st) = \text{hyp}\,(s) + \text{hyp}\,(t)$$

without using the *formula* for "change of variable" in integration. The method incidentally shows the graphical equivalent of the simple change of variable $z = tx$ in integration.

The diagram represents the graph of $y=1/x$; $OP=x$, $PQ=\Delta x$, $OK=tx$, $KL=t.\Delta x$.



Then
$$\begin{aligned}\text{hyp } (s) &= \int_1^s dx/x \\ &= \text{area } ABCD \\ &= \lim \Sigma PQRS \\ &= \lim \sum_{x=1}^{s} \Delta x/x.\end{aligned}$$
$$\begin{aligned}\text{hyp } (st) - \text{hyp } (t) &= \text{area } EFGH \\ &= \lim \Sigma KLMN \\ &= \lim \sum_{tx=t}^{ts} \frac{t.\Delta x}{tx} \\ &= \lim \sum_{x=1}^{s} \Delta x/x.\end{aligned}$$

Thus the two functions on the left-hand side are equal and the theorem is proved. F. J. TONGUE.

1997. *Elementary derivation of the Euler-Maclaurin summation formula.* By Taylor's theorem,
$$\phi(b) = \phi(a) + \phi'(a).h + \phi''(a).h^2/2! + \dots,$$
$$\phi(a) = \phi(b) - \phi'(b)h + \phi''(b).h^2/2! - \dots,$$
where $h=b-a$. Hence
$$2\{\phi(b)-\phi(a)\} = \{\phi'(b)+\phi'(a)\}h - \{\phi''(b)-\phi''(a)\}h^2/2! + \{\phi'''(b)+\phi'''(a)\}h^3/3! - \dots,$$
and therefore, if we write S_n for $\phi^n(b)+\phi^n(a)$ and D_n for $\phi^n(b)-\phi^n(a)$,
$$2D_0 = S_1h - D_2h^2/2! + S_3h^3/3! - \dots. \qquad \dots\dots\dots\

$$2D_0 = S_1h - D_2h^2/2! + (2D_2 + D_4h^2/2! - S_3h^3/3! + \dots)(h^2/3!) - D_4h^4/4! + \dots$$
$$= S_1h - D_2h^2/6 + D_4h^4/24 - S_5h^5/36 + \dots$$
$$= S_1h - D_2h^2/6 + D_4h^4/72 - \dots .$$

In this way we can express S_1 in terms of $D_0, D_2, \dots$. Hence if $\phi(n) = \int f(x)dx$,

$$D_n = \phi^n(b) - \phi^n(a) = f^{n-1}(b) - f^{n-1}(a),$$

and
$$S_1 = \phi'(b) + \phi'(a) = f(b) + f(a).$$

Therefore we find

$$h\{f(b)+f(a)\} = 2\int_a^b f(x)dx - \{f'(b) - f'(a)\}\frac{h^2}{6} + \{f'''(b) - f'''(a)\}\frac{h^4}{72} - \dots \dots \text{(ii)}$$

Applying this formula for each interval h from a to $a+nh$ and adding we get

$$2h\sum_{r=0}^{n} f(a+rh) - h\{f(a+nh)+f(a)\}$$
$$= 2\int_a^{a+nh} f(x)ax - \{f'(a+nh) - f'(a)\}\frac{h^2}{6} + \{f'''(a+nh) - f'''(a)\}\frac{h^4}{72} + \dots ,$$

which is the Euler-Maclaurin formula.

Alternatively, writing E for the operation which changes ϕ^n to ϕ^{n+1} in $\phi^n(a)$ and $\phi^n(b)$, we have from (i)

$$\{2 + E^2h^2/2! + E^4h^4/4! + \dots\} D_0 = \{Eh + E^3h^3/3! + \dots\} S_0$$

whence
$$S_0 = \frac{1+\cosh Eh}{\sinh Eh} \cdot D_0$$
$$= \{\coth \tfrac{1}{2}Eh\} D_0$$
$$= \frac{1}{\frac{1}{2}Eh} \cdot \Sigma(-)^n \frac{B_{2n}E^{2n}h^{2n}}{(2n)!} \cdot D_0.$$

Thus
$$\tfrac{1}{2}hS_1 = \tfrac{1}{2}Eh \,.\, S_0 = \Sigma(-)^n B_{2n}h^{2n}D_{2n}/(2n)!$$

or
$$h\{f(b)+f(a)\} = 2D_0 - 2B_2h^2D_2/2! + 2B_4h^4D_4/4! - \dots ,$$

which is the same as (ii). H. V. LOWRY.

1998. *Euler's transformation of alternating series.*

The interesting Note 1855 by Mr. Paradine on the calculation of the sum of a slowly converging alternating series made me reconsider Euler's formula for the sum of such a series. The following proof is, I think, more elementary than those usually given.

Let the series be $a_1 - a_2 + a_3 - \dots$, S_n its sum to n terms and S its sum to infinity. Let

$$f(n) = S_n + (-)^n\left\{\frac{a_{n+1}}{2} - \frac{\Delta a_{n+1}}{2^2} + \dots + \frac{(-\Delta)^{r-1}a_{n+1}}{2^{r+1}}\right\}$$

$$= S_n + (-)^n\left\{\frac{1-(-\frac{1}{2}\Delta)^r}{2+\Delta}\right\} a_{n+1},$$

where r is an integer. The form of this function is suggested by the operational form for the remainder after n terms, namely

$$\frac{(-)^n}{2+\Delta} \cdot a_{n+1}.$$

Then it is easily shown that

$$f(n+1) - f(n) = (-)^n(-\tfrac{1}{2}\Delta)^r a_{n+1} \quad \dots\dots\dots\dots\dots \text{(i)}$$
$$f(n+2) - f(n) = (-)^n(-\tfrac{1}{2}\Delta)^{r+1} a_{n+1} \quad \dots\dots\dots\dots\dots \text{(ii)}$$

Now, clearly, $f(n) \to S$ as $n \to \infty$. Moreover, provided that $(-\Delta)^r a_{n+1}$ is

positive for all r, that is, provided $\Sigma(-)^{n-1}a_n$ is fully monotone, (ii) shows that for n even the values of $f(n)$ form an increasing sequence and for n odd they form a decreasing sequence. Hence S lies between $f(n)$ and $f(n+1)$, but (i) shows that $f(n+1)-f(n)$ is of the order $2^{-r}\,|\,\Delta^r a_{n+1}\,|$ and therefore that

$$|\,S-f(n)\,|<2^{-r}\,|\,\Delta^r a_{n+1}\,|.$$

It is easily shown that when the series is fully monotone, $|\,2^{-r}\Delta^r a_{n+1}\,|$ decreases with r and hence $f(n)$ is a better approximation to S than S_n is, and the approximation is improved by increasing the value of r.

A still more elementary approach is to assume

$$f(n)=S_n+(-)^n\{\lambda_1 a_{n+1}+\lambda_2 a_{n+2}+\ldots+\lambda_r a_{n+r}\}$$

and to choose $\lambda_1, \ldots, \lambda_r$ so that $f(n+1)-f(n)$ is of order $\Delta^r a_{n+1}$, that is,

$$f(n+1)-f(n)=k\left\{1+rE+\frac{r(r-1)}{2!}E^2+\ldots+E^r\right\}a_{n+1}$$

where k is a constant. For example, when $r=2$, we have

$$f(n+1)-f(n)=(-)^n\{(1-\lambda_1)a_{n+1}-(\lambda_1+\lambda_2)a_{n+2}-\lambda_2 a_{n+3}\}$$

whence $\lambda_2=-1+\lambda_1$ and $\lambda_1+\lambda_2=-2\lambda_2$, so that $\lambda_1=\frac{3}{4}$, $\lambda_2=-\frac{1}{4}$.

Hence $\qquad f(n)=S_n+(-)^n\ (\frac{3}{4}a_{n+1}-\frac{1}{4}a_{n+2})$,

which is the same as

$$f(n)=S_n+(-)^n\left(\frac{a_{n+1}}{2}-\frac{\Delta a_{n+1}}{2^2}\right).$$

H. V. LOWRY.

1999. *The projection of a conic into a circle.*

1. There are three common elementary definitions of a conic, namely that a conic is :

(*a*) a plane section of a circular cone (or, what is the same thing, a conical projection of a circle) ;

(*b*) a plane locus defined by the focus-directrix property $SP=e\,.\,PM$ (and expressed algebraically by one or other of three standard equations) ;

(*c*) a plane curve whose equation referred to rectangular Cartesian axes is the general equation of the second degree,

$$S\equiv ax^2+2hxy+by^2+2gx+2fy+c=0,\ \text{ with } \Delta\neq 0.$$

We may name the conics so defined the Cone-section conic, the Focus-directrix conic and the General-equation conic respectively.

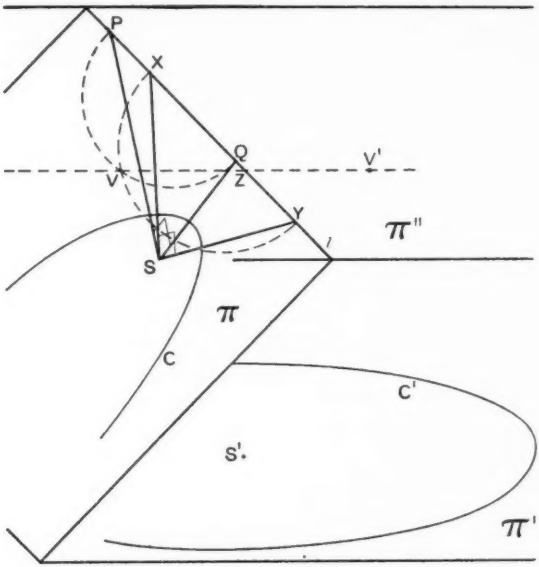
2. At some stage, we wish to show the equivalence of these definitions. Once the translation and rotation of axes has been dealt with, we can reduce the equation (*c*) to one of the standard equations of the Focus-directrix conic (including a circle as an ellipse with eccentricity zero). Hence we show that the Focus-directrix conic and the General-equation conic are one and the same.

3. We may use Dandelin's construction of the inscribed sphere to show that a plane section of a *right* circular cone has the focus-directrix property ; but the method fails for a section of a general circular cone (Sommerville, *Analytical Conics*, pp. 36 A and B.

4. It is shown in Durell's *Projective Geometry* (p. 41) that conical projection is a collineation, so that the projection of a circle is a locus of the second degree ; that is, that a Cone-section conic is also a General-equation conic, and hence, by (2) above, a Focus-directrix conic.

5. The interesting step is the converse one, namely to show that a Focus-directrix conic is also a Cone-section conic, and so to complete the equivalence of the three definitions. Given a Focus-directrix conic we wish to show how to project it into a circle.

6. If we accept an " imaginary " vertex of projection, that is, if we solve the problem analytically, we may follow Durell, *Projective Geometry*, p. 76. A " real " construction, however, seems most desirable ; and such a construction, which does not seem to be widely known, is the following :



A conic C with focus S and directrix l, lying in a plane π, is given. Let π'' be any plane (other than π) through l, and π' any plane parallel to π''. Then if we project π from any point V (not on l) of π'' on to π', l is the vanishing line.

By Durell's method, referred to in (4), we know that the projection is a Focus-directrix conic. Also, since C does not meet the directrix l, the projection C' contains only finite points and so is an ellipse.

We now choose V in π'' so as to ensure that the ellipse C' is a circle.

Let any two pairs of perpendicular lines in π through S meet l in P, Q, X, Y. Then P and Q are separated by X or Y. Hence circles in the plane π'' on diameters PQ and XY intersect in real points V, V', and we take V as vertex of projection. Now, the pairs of perpendicular lines through the focus S are each conjugate with respect to C. They project into pairs of conjugate lines with respect to C' through S', the projection of S. Moreover, since S is the pole of l with respect to C, S' is the pole of the line at infinity with respect to C', that is, the centre of C'.

The pairs of perpendicular lines therefore project into pairs of conjugate diameters of C'. Further, by construction of V, they remain perpendicular pairs. Hence C' has two pairs of perpendicular conjugate diameters. Hence C' is a circle.

(It will be noted that V and V' lie in π'' in the plane through the major axis of C perpendicular to l, and are such that $VZ = V'Z = SZ$.) P. GANT.

2000. *On differentials.*

The very interesting article on this subject in the *Gazette* of December, 1945 (XXIX, p. 193) has set one thinking again, on a question of fundamental importance. It should be noted that the discussion in the Association goes back beyond 1936, to an article in XV, p. 401 (July, 1931), which was followed by a valuable exchange of views in XVI, p. 5 (February, 1932), and then by a general M.A. Annual Meeting discussion in 1934 (XVIII, p. 68) led by Professor Temple. The essential points I want to make are in an article in XIX, p. 79 (May, 1935), and need not be repeated here. I concur emphatically in the objection to defining the differential dx by the arbitrary infinitesimal * δx (with the implicit distinction between "independent" and "dependent" variables): a definition I have never found to be of the slightest actual importance. But the suggested "new viewpoint"—though there is much more sense to it—has the even more serious drawback that it takes dx right out of the realm of the infinitesimal, *e.g.* for the simplest choice of parameter, $dx = 1$; and that appears to be fatal to its genuine usefulness. The somewhat forced arguments of pp. 197-9 are significant; and the climax that $d^2y \neq y''.dx^2$, etc., is an extraordinarily serious consequence of the "new definition of differentials". The article impresses me as a valiant attempt to establish an elementary *quantitative* theory of differentials; but leaves me the more convinced that no such theory is either feasible or necessary. The *non-quantitative* elementary proposition † that "the differential symbol 'd', in all its legitimate uses, may be interpreted to mean '$\lim \delta$'—and, similarly, the symbol '$\int$' to mean '$\lim \Sigma$'"—expresses, I believe, the true relation of the differential du to the increment infinitesimal δu, etc., and it is not in serious conflict with the somewhat abstruse theory presented by Professor Temple. (But why need there be any "flight from" limits?)

One other point: there is some danger that undue emphasis (at an early stage) on the unicursal specification of functional relation between x and y may tend to obscure the basic importance of the elementary case in which y is an explicit one-valued function of x (the case to which both the relations $x = \phi(t)$, $y = \psi(t)$ of the unicursal specification essentially belong). The analogies stated on p. 195 are interesting, but it may be questioned whether either is really applicable.‡

I wish it to be clear that the criticisms I have felt impelled to offer are, in fact, a tribute to the value of the article to one of the most important and instructive discussions our Association has ever staged. D. K. PICKEN.

2001. *On the calculation of radii of gyration.*

The average student does not know by heart the values of the radii of gyration associated with triangles. Further, he generally regards as tedious the usual method of calculation by subdivision of the triangle into strips. It is suggested that the following interesting and instructive method of obtaining radii of gyration associated with triangles and parallelograms would be appreciated by the student.

Let M denote the mass of any lamina under consideration. The following properties of moments of inertia and radii of gyration are required:

* See second foot-note, XIX, p. 80.

† See XIX, top of pages 81, 83.

‡ The essential point of the analogy (*a*) is a very definite intrinsic theoretical system of "derived units" of geometrical and physical quantities (see XXVI, especially pp. 91, 95, May, 1942). As to the analogy (*b*), no such arbitrary choice of parameter (as of the origin of the vectors) is, in fact, available (none at all, may be).

(*a*) moments of inertia are additive ;
(*b*) radii of gyration associated with similarly situated axes of similar plane figures are proportional to the square roots of the areas ;
(*c*) the parallel axes theorem, that $\bar{k}^2 = k^2 + h^2$, where k refers to an axis through the mass centre, $\bar{k}$ to a parallel axis, and h is the perpendicular distance between these axes.

I. *To calculate the radius of gyration k about an axis through the mass centre O, and perpendicular to the plane, of the parallelogram with sides of length* 2*a*, 2*b*.

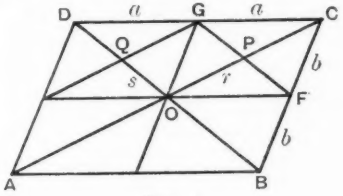


FIG. 1.

Divide the parallelogram $ABCD$ into four similar and equal parallelograms by parallels to the sides through O. By property (*b*) the radius of gyration of the parallelogram $OFCG$ about an axis through P perpendicular to its plane is $\frac{1}{2}k$. Using (*a*) and (*c*), the following equation is immediately obtained :

$$2[\tfrac{1}{4}M\{(\tfrac{1}{2}k)^2 + r^2\}] + 2[\tfrac{1}{4}M\{(\tfrac{1}{2}k)^2 + s^2\}] = Mk^2,$$

where $OP = r$ and $OQ = s$. From the geometry of the figure,

$$r^2 + s^2 = 2\{(\tfrac{1}{2}a)^2 + (\tfrac{1}{2}b)^2\}.$$

Combination of these two equations leads to the result

$$k = (a^2 + b^2)^{\frac{1}{2}}/\sqrt{3} \quad \text{.....(i)}$$

II. *To calculate the radius of gyration k about an axis through the mass centre O, and perpendicular to the plane, of the equilateral triangle with sides of length* 2*a*.

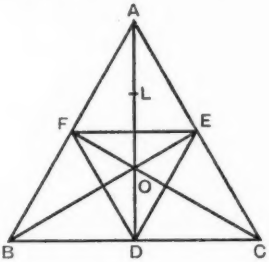


FIG. 2.

Divide the triangle into four equal equilateral triangles by joining the mid-points of the sides, as in Fig. 2. Then let L be the mass centre of the triangle AFE ; hence $OL = a/\sqrt{3}$. By property (*b*), the radius of gyration of the triangle AFE about an axis through L perpendicular to its plane is $\frac{1}{2}k$. Using (*a*) and (*c*), the following equation is immediately obtained :

$$3[\tfrac{1}{4}M\{(\tfrac{1}{2}k)^2 + (a/\sqrt{3})^2\}] + \tfrac{1}{4}M(\tfrac{1}{2}k)^2 = Mk^2,$$

leading immediately to

$$k = a/\sqrt{3}. \quad \text{.....(ii)}$$

III. *To calculate the radius of gyration k about an axis through the mass centre O, and perpendicular to the plane, of the triangle with sides of lengths, $2a$, $2b$, $2c$.*

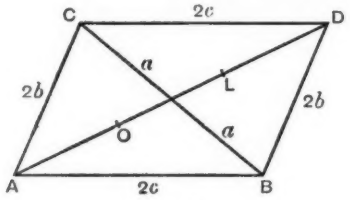


FIG. 3.

This problem could be solved by the method of section II. It is simpler to proceed in the following way. Complete the parallelogram $ABDC$ as in Fig. 3. Let L be the mass centre of the triangle BCD. Using the result (i) and properties (a) and (c), the following equation is easily obtained :

$$2M(k^2 + r^2) = (2M)(b^2 + c^2)/3,$$

where $OL = r$. From the geometry of the figure,

$$4c^2 + 4b^2 = 2(9r^2 + a^2).$$

Combination of these two equations leads to the result

$$k = (a^2 + b^2 + c^2)^{\frac{1}{2}}/3.$$

Similar methods can be used to calculate radii of gyration about axes lying in the plane of the lamina. B. SPAIN.

2002. *Proofs of two trigonometrical identities.*

In the triangle ABC, $BN : NC = m : n$. Then

$$(m+n)\cot\theta = n\cot B - m\cot C \quad \text{(Fig. 1)},$$
$$(m+n)\cot\theta = m\cot\beta - n\cot\gamma \quad \text{(Fig. 2)}.$$

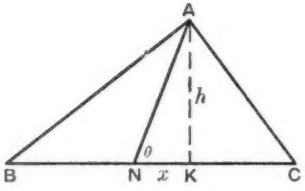


FIG. 1.

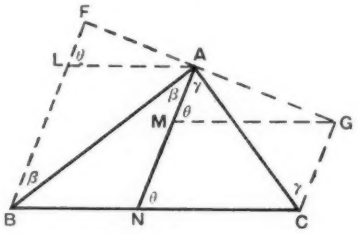


FIG. 2.

For, in Fig. 1, $(m+n)\cot\theta = m(x/h) + n(x/h)$

$= m(NC - KC)/h + n(BK - BN)/h$

$= n(BK/h) - m(KC/h)$, since $m \,.\, NC = n \,.\, NB$

$= n\cot B - m\cot C.$

In Fig. 2, FAG is perpendicular to BF, NA, CG.

$$AN = BF - LF = AF \cot \beta - AF \cot \theta,$$

and

$$AN = GC + AM = AG \cot \gamma + AG \cot \theta.$$

Since

$$AF/AG = BN/NC = m/n,$$

$$(m+n) \cot \theta = m \cot \beta - n \cot \gamma.$$

F. L. GOODWIN.

2003. *Recurrence formulae for Legendre functions.*

1. We establish in this note the well-known group of formulae connecting Legendre functions of different degrees, by a method more natural and straightforward than (though running on the same lines as) that given in Whittaker and Watson, *Modern Analysis*. Our method consists in adopting the convenient notation

$$\frac{(t^2-1)^{n+1}}{(t-z)^{n+2}} \equiv [n+1, n+2]$$

and

$$\frac{t(t^2-1)^n}{(t-z)^{n+1}} \equiv t[n, n+1],$$

differentiating with respect to t and interpreting the results in terms of Legendre functions by means of the formulae

$$P_n(z) = \frac{1}{2^{n+1}\pi i} \int_C [n, n+1]\, dt, \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$$P_n'(z) = \frac{n+1}{2^{n+1}\pi i} \int_C [n, n+2]\, dt. \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Now,

$$\frac{d}{dt}[n+1, n+2] = 2t(n+1)[n, n+2] - (n+2)[n+1, n+3]$$

$$= 2(n+1)[n, n+1] + 2z(n+1)[n, n+2] - (n+2)[n+1, n+3]$$

writing $(t - z + z)$ for t.

Integrating round a suitable contour C, we have

$$2(n+1) \int_C [n, n+1]\, dt + 2z(n+1) \int_C [n, n+2]\, dt = (n+2) \int_C [n+1, n+3]\, dt.$$

Dropping the common factor $2^{n+2}\pi i$, we get

$$(n+1)P_n(z) + zP_n'(z) = P_{n+1}'(z) \text{ by using (i) and (ii).}$$

Again

$$\frac{d}{dt}\, t[n, n+1] = -t(n+1)[n, n+2] + 2t^2 n[n-1, n+1] + [n, n+1]$$

$$= -(t - z + z)(n+1)[n, n+2] + 2(t^2 - 1 + 1)n[n-1, n+1] + [n, n+1]$$

$$= n[n, n+1] - z(n+1)[n, n+2] + 2n[n-1, n+1].$$

Integrating round a suitable contour C, we have

$$n \int_C [n, n+1]\, dt - z(n+1) \int_C [n, n+2]\, dt + 2n \int_C [n-1, n+1]\, dt = 0.$$

Dropping the common factor $2^{n+1}\pi i$, we get

$$nP_n(z) - zP_n'(z) + P_{n-1}'(z) = 0 \text{ by using (i) and (ii).}$$

We thus get

$$P_{n+1}'(z) - zP_n'(z) = (n+1)P_n(z) \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(a)}$$

$$zP_n'(z) - P_{n-1}'(z) = nP_n(z) \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(b)}$$

Adding, we get,

$$P'_{n+1}(z) - P'_{n-1}(z) = (2n+1)P_n(z). \quad \text{.....(c)}$$

Changing n to $n-1$ in (a) and solving for $P'_n(z)$ and $P'_{n-1}(z)$, we get

$$(z^2-1)P'_n(z) = nzP_n(z) - nP_{n-1}(z), \quad \text{.....(d)}$$

and

$$(z^2-1)P'_{n-1}(z) = nP_n(z) - nzP_{n-1}(z). \quad \text{.....(e)}$$

Changing n to $n+1$ in (e) and subtracting (d), we have

$$(n+1)P_{n+1}(z) - (2n+1)zP_n(z) + nP_{n-1}(z) = 0. \quad \text{.....(f)}$$

Differentiating (d) with respect to z, we get

$$(z^2-1)P''_n(z) + 2zP'_n(z) = nzP'_n(z) + nP_n(z) - nP'_{n-1}(z)$$
$$= n(n+1)P_n(z) \quad \text{using (b).}$$

Hence

$$(1-z^2)P''_n(z) - 2zP'_n(z) + n(n+1)P_n(z) = 0,$$

which is the well-known Legendre's differential equation.

Thus it seems that (a) and (b) are really the fundamental recurrence formulae from which all others, viz. (c), (d), (e) and (f) are obtained by mere algebraic processes (involving no differentiation). In fact, from any two of the six formulae (a), (b), (c), (d), (e) and (f) we can derive the other four by algebraic manipulations.

2. Whittaker and Watson have suggested alternative methods of proof applicable to the case when n is a positive integer, only for the identities (b) and (f) above. We wish to point out that the formulae (a), (c) and (d) can be derived directly by equating coefficients of powers of h from the following obvious identities :

$$\frac{1-zh}{h}\frac{\partial V}{\partial z} - h\frac{\partial V}{\partial h} = V,$$

$$\frac{1-h^2}{h}\frac{\partial V}{\partial z} - 2h\frac{\partial V}{\partial h} = V,$$

$$\frac{1-z^2}{h}\frac{\partial V}{\partial z} + (z-h)\frac{\partial V}{\partial h} = V,$$

where $V = (1-2zh+h^2)^{-\frac{1}{2}}$.

A. K. SRINIVASAN.

CORRESPONDENCE.

W. K. CLIFFORD.

To the Editor of the *Mathematical Gazette*.

SIR,—I protest most strongly against your suggestion (*Math. Gazette*, No. 294, p. 120) that Mallock's " portrait " in *The New Republic* is only a distortion and not a falsification of Clifford's attitude. Anyone who compares the *Lectures and Essays* with *The New Republic* can easily see the fundamental difference between Clifford and Mr. Saunders. Yours, etc., F. E. CAVE.

[The precise point at which distortion becomes falsification is not easy to determine, but I cannot see in Mallock's book anything more than that distortion which is permissible and perhaps essential in a satire. If my remark and Miss Cave's comment will send readers to the *Lectures and Essays* and to *The New Republic*, as well as to the new edition of *The Common Sense of the Exact Sciences*, I shall be content, whatever views they take of " Mr. Saunders ".—T. A. A. B.]

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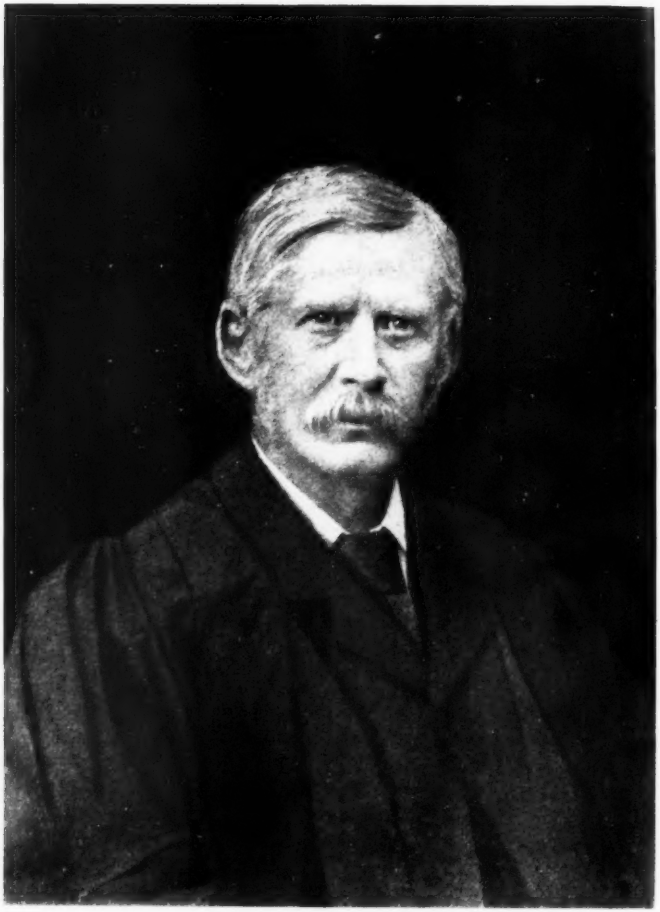

RAWDON LEVETT